Combinatorial Principles and $\aleph_k$-free Modules

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A mi esposa Marcela.

Contents

Introduction

As part of his PhD-thesis at Cambridge University, A.L.S. Corner obtained the following

Theorem 1. *Every countable reduced torsion-free ring R with 1 is isomorphic to the endomorphism ring* $\operatorname{End} G$ *of a countable reduced torsion-free group G.*

This powerful theorem (in [**1**]) is very important since it permits solving various problems by choosing different rings for R. For example, this allows to give a negative answer to Kaplansky's Test Problems. Years later this result was extended by A.L.S. Corner and R. Göbel in [**2**] to modules over rings of uncountable size using a combinatorial principle by S. Shelah called the "Black Box". This also allows to construct E-rings which are now widely used in algebraic topology. It was already noted in [**2**] that the constructed modules were $\aleph_1$-free (i.e. every $< \aleph_1$-generated submodule is contained in a free submodule) but it was clear that they were not $\aleph_2$-free. Shortly after another paper by M. Dugas and R. Göbel [**5**] discussed separable groups, in which every finite subset is contained in a free direct summand of the group. In the case of torsion-free abelian groups this means that the group is a pure subgroup of a Cartesian product of copies of the integers. The main result in this paper can be summarized in the following

THEOREM 2. *For every infinite cardinal* λ *such that* $\lambda^{\aleph_0} = \lambda$ *there exists a separable* $\aleph_1$*-free abelian group* G *with* $\operatorname{End} G = \mathbb{Z} \oplus \operatorname{Fin} G$, *where* $\operatorname{Fin} G$ *is the ideal generated by all the endomorphisms of* G *whose images have finite rank.*

Already in the first half of the last century Baer, Specker ([**10**], vol. 1, p. 94) and Łoś ([**10**], vol. 2, p. 161) observed that there are complicated $\aleph_1$-free groups that are not free. One famous example of such groups is the Baer-Specker group $\prod_{n<\omega} \mathbb{Z}e_n$. G. Nöbeling and G.M. Bergman ([**10**], vol. 2, p. 173, 174) observed that these groups contain large "canonical" free subgroups. It is natural to ask if we can strengthen the theorem from [**2**] to obtain $\aleph_k$-free modules, namely, can we find complicated $\aleph_k$-free modules which are not free for $k > 1$? Classical work in this direction was done by P. Griffith [**16**], P. Hill [**17**], P. Eklof and S. Shelah [**9**]. While this can be relatively easily done assuming additional set-theoretic assumptions such as the axiom of constructibility $V = L$ (see Dugas, Göbel [**4**]), the construction of such modules within ZFC without extra axioms is a complicated target.

Göbel and Shelah [**12**] follow this approach by using a modification of the Black Box in their proof:

THEOREM 3. *For every* $k < \omega$ *there exists an* $\aleph_k$*-free abelian group* G *of size* $\beth_k$ *such that* $\operatorname{Hom}(G, \mathbb{Z}) = 0$.

In 2010 R. Göbel, D. Herden and S. Shelah [**11**] developed a new, more powerful version of the Black Box principle, called the $\aleph_k$-Black Box, which allowed them to prove the following

THEOREM 4. *Let R be a p-cotorsion-free domain and A an R-algebra with free R-module A_R and $|A| \leq \mu$. If $\lambda = \beth_k^+(\mu)$ for some positive integer k, where $\beth_0^+(\mu) = \mu$ and $\beth_{n+1}^+(\mu) = \left(2^{\beth_n^+(\mu)}\right)^+$, which is the successor cardinal of the powerset of $\beth_n^+(\mu)$, then we can construct an $\aleph_k$-free A-module G of cardinality λ with R-endomorphism algebra* $\operatorname{End}_R G = A$.

The two main results of this work deal with the construction of separable $\aleph_k$-free abelian groups following the approach of Theorem 3 and Theorem 4. It is therefore natural to keep the notation as similar as possible to the ones used in these papers. We cannot expect these groups to have a trivial dual $\operatorname{Hom}(G, \mathbb{Z})$ or satisfy $\operatorname{End} G = A$, because separable groups embed into cartesian products $\prod_{i \in I} \mathbb{Z}e_i$. The projections induced by these products cannot be eliminated and require to consider the ideal $\operatorname{Fin} G$.

The first result of this work is the construction of a separable $\aleph_k$-free abelian group G of size $\beth_k$ with no epimorphisms onto $\bigoplus_{n<\omega} \mathbb{Z}e_n$, and is based on methods used in the proof of Theorem 3:

THEOREM 5. *For every $k > 1$ there exists a separable $\aleph_k$-free abelian group G of size $\beth_k$ with no epimorphisms onto $\bigoplus_{n<\omega} \mathbb{Z}e_n$.*

The second result shows the existence of an $\aleph_k$-free A-module G of size $\beth_k^+(\mu)$ which is separable as an abelian group and satisfies $\operatorname{End} G = A \oplus \operatorname{Fin} G$, where A is a given ring ($\mathbb{Z}$-algebra) with $|A| \leq \mu$ and free additive structure $A^+ = \bigoplus_{\alpha<\kappa} \mathbb{Z}e_\alpha$. Here we follow [**11**]:

THEOREM 6. *For every $k > 1$ there exist arbitrarily large $\aleph_k$-free A-modules G, which are separable as abelian groups, such that* $\operatorname{End} G = A \oplus \operatorname{Fin} G$.

This work is presented in two chapters, one for each of these main results. The first section of Chapter 1 introduces the basic notations and definitions, in particular the definition of the basic sets Λ and Λ_*. We also mention the set-theoretic version of the new Black Box. Section 2 presents the new Black Box in an algebraic context and constructs separable $\aleph_k$-free groups G satisfying $\operatorname{Hom}(G, H) = \operatorname{Fin}(G, H)$ for H a free abelian group. In Section 3 we briefly discuss the slenderness of $\aleph_k$-free groups.

The first section of Chapter 2 considers a ring A with free additive structure. After some slight modifications to the set-theoretic notions and definitions of Chapter 1, we construct $\aleph_k$-free A-modules, which are separable abelian groups, by means of pairs of subsets of our basic sets. Section 2 introduces another kind of A-modules with important freeness properties, which are defined by means of triples of subsets of these basic sets. The focus of Section 3 is the Step Lemma, which is the central piece of the final construction. Finally, Section 4 presents the main construction with the help of the Strong Black Box and a result based on [**5**].

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List of Symbols

CHAPTER 1

Separable $\aleph_k$-free Groups with Small Dual

1. Set-theoretic Preliminaries

Let us begin by introducing some notation:

(1) Functions will be written on the right side of their argument, so if f is a function with domain A and $a \in A$, then the image of a under f will be written as af.

(2) ${}^{\omega}\lambda$ denotes the set of all functions $\tau : \omega \to \lambda$, while ${}^{\omega\uparrow}\lambda$ is the subset of ${}^{\omega}\lambda$ consisting of all *order preserving* functions $\eta : \omega \to \lambda$, namely

$$ {}^{\omega\uparrow}\lambda = \{\, \eta : \omega \to \lambda \mid m\eta < n\eta \text{ for } m < n \,\}. $$

Similarly, ${}^{\omega>}\lambda$ denotes the set of all functions $\sigma : n \to \lambda$ with $n < \omega$, while ${}^{\omega\uparrow>}\lambda$ is the subset of ${}^{\omega>}\lambda$ consisting of all *order preserving* functions $\eta : n \to \lambda$ with $n < \omega$.

(3) If $f : A \to {}^{B}C$, i.e. af is a function for all $a \in A$, then we write f_a instead of af.

(4) If A is a set and κ is a cardinal, then $[\,A\,]^{\leq\kappa}$ denotes the set of all $X \subseteq A$ such that $|\,X\,| \leq \kappa$. Analogously we define $[\,A\,]^{<\kappa}$ and $[\,A\,]^{\kappa} = [\,A\,]^{=\kappa}$.

(5) If $\alpha \leq \gamma$ are ordinals, we write $[\,\alpha, \gamma\,] = \{\, \beta \mid \alpha \leq \beta \leq \gamma \,\}$, $(\,\alpha, \gamma\,) = \{\, \beta \mid \alpha < \beta < \gamma \,\}$ and $[\,\alpha, \gamma\,) = \{\, \beta \mid \alpha \leq \beta < \gamma \,\}$.

(6) The expression $H \leq G$ denotes that H is a subgroup a G.

(7) Let $\{ A_i \mid i \in [1,m] \}$ and $\{ B_i \mid i \in [1,n] \}$ be finite families of sets, $A = A_1 \times \cdots \times A_m$ and $B = B_1 \times \cdots \times B_n$. If $a = (a_1, \ldots, a_m) \in A$ and $b = (b_1, \ldots, b_n) \in B$, we write $a \wedge b = (a_1, \ldots, a_m, b_1, \ldots, b_n)$. If $X \subseteq A$ and $Y \subseteq B$, we write $X \wedge Y = \{ a \wedge b \mid a \in X, b \in Y \}$.

Let $k > 1$ be fixed for the rest of this work. Given a finite sequence of infinite cardinals $\langle \lambda_1, \ldots, \lambda_k \rangle$ and $i, j \in [1,k]$ with $i \leq j$, we construct the set

$$\Lambda^{[i,j]} = {}^{\omega\uparrow}\lambda_i \times {}^{\omega\uparrow}\lambda_{i+1} \times \cdots \times {}^{\omega\uparrow}\lambda_{j-1} \times {}^{\omega\uparrow}\lambda_j.$$

For $\Lambda^{[1,k]}$ we simply write Λ. Moreover, for $i < j$, define $\Lambda_i^{[i,j]} = {}^{\omega\uparrow>}\lambda_i \wedge \Lambda^{[i+1,j]}$, $\Lambda_j^{[i,j]} = \Lambda^{[i,j-1]} \wedge {}^{\omega\uparrow>}\lambda_j$ and

$$\Lambda_m^{[i,j]} = \Lambda^{[i,m-1]} \wedge {}^{\omega\uparrow>}\lambda_m \wedge \Lambda^{[m+1,j]},$$

for all $m \in (i,j)$. Put

$$\Lambda_*^{[i,j]} = \dot{\bigcup_{m \in [i,j]}} \Lambda_m^{[i,j]}.$$

For $i = 1$ and $j = k$ we simply write Λ_* and Λ_m for all $m \in [1,k]$.

Definition 7. (1) If $\eta \in {}^{\omega\uparrow}\lambda_m$, then the *support* of η is the set

$$[\eta] = \{ \eta \restriction n \mid n < \omega \}.$$

(2) If $\nu \in {}^{\omega>\uparrow}\lambda_m$ with $\operatorname{dom} \nu = n$, then the *support* of ν is the set

$$[\nu] = \{ \nu \restriction \ell \mid \ell \leq n \}.$$

(3) If $\overline{\eta} = (\eta_i, \ldots, \eta_j) \in \Lambda^{[i,j]}$, $m \in [\,i, j\,]$ and $n < \omega$, then $\overline{\eta} \upharpoonleft \langle m, n \rangle$ denotes the element of $\Lambda_m^{[i,j]}$ obtained from $\overline{\eta}$ by replacing its component η_m by $\eta_m \upharpoonright n$, i.e.

$$\overline{\eta} \upharpoonleft \langle m, n \rangle = (\eta_i, \ldots, \eta_{m-1},\, \eta_m \upharpoonright n,\, \eta_{m+1} \ldots, \eta_j).$$

(4) For every $\overline{\eta} \in \Lambda^{[i,j]}$, $m \in [\,i, j\,]$ and $n < \omega$ consider the sets

$$[\,\overline{\eta} \upharpoonleft m\,]_n = \{\, \overline{\eta} \upharpoonleft \langle m, n' \rangle \mid n' \in [\,n, \omega)\,\}$$

and

$$[\,\overline{\eta}\,]_n = \dot{\bigcup_{m \in [\,i,j\,]}} [\,\overline{\eta} \upharpoonleft m\,]_n$$

If $n = 0$, then we simply drop the subindex and write $[\,\overline{\eta} \upharpoonleft m\,]$ and $[\,\overline{\eta}\,]$ instead. The set $[\,\overline{\eta}\,]$ is called the *support* of $\overline{\eta}$.

Definition 8. Define the *norm function* $\|\cdot\| : \Lambda \dot{\cup} \Lambda_* \to \lambda_k$ as

$$\|\,\overline{\eta}\,\| = \sup \operatorname{Im} \eta_k.$$

For $X \subseteq \Lambda \dot{\cup} \Lambda_*$, we naturally define

$$\|\,X\,\| = \sup_{\overline{\eta} \in X} \|\,\overline{\eta}\,\|\,.$$

Definition 9. A function $F : \Lambda \to [\,\Lambda_*\,]^{\leq \aleph_0}$ is *regressive* if $\|\,\overline{\eta} F\,\| < 0\eta_k$ for all $\overline{\eta} \in \Lambda$.

The first step towards the proof of the main result concerning $\aleph_k$-freeness is the so called Freeness-Proposition, which allows us to enumerate subsets of Λ in such a "clever" way that we can prove linearly independence in the constructed groups. First we have to deal with the notion of coordinatewise-closedness of filtrations, which will appear in the proof of the proposition.

LEMMA 10. *Let $F : \Lambda \to [\Lambda_*]^{\leq \aleph_0}$ be any map, $f \in [1, k]$ and $\Omega \in [\Lambda]^{\aleph_f}$ together with an $\aleph_f$-filtration $\{ \mho^\alpha \mid \alpha < \omega_f \}$ of Ω such that $\mho^0 = \emptyset$ and $|\, \mho^{\alpha+1} \setminus \mho^\alpha \,| = \aleph_{f-1}$ for all $\alpha < \omega_f$. Then it is possible to construct a coordinatewise-closed $\aleph_f$-filtration $\{ \Omega^\alpha \mid \alpha < \omega_f \}$ of Ω, meaning that for all $\overline{\eta} \in \Omega^{\alpha+1}$, if there exist $\overline{\eta}'$, $\overline{\eta}'' \in \Omega^\alpha$ such that*

$$\{ \eta_m \mid m \in [1, k] \} \subseteq \{ \eta'_m, \eta''_m \mid m \in [1, k] \} \cup \{ \nu_m \mid \overline{\nu} \in \overline{\eta}'F \cup \overline{\eta}''F, m \in [1, k] \},$$

then $\overline{\eta} \in \Omega_\alpha$. This $\aleph_f$-filtration also satisfies $\Omega^0 = \emptyset$ and $|\, \Omega^{\alpha+1} \setminus \Omega^\alpha \,| = \aleph_{f-1}$ for all $\alpha < \omega_f$.

PROOF. First suppose that we have constructed a coordinatewise-closed $\aleph_f$-filtration $\{ \Omega^\alpha \mid \alpha < \gamma \}$ of Ω up to some limit ordinal $\gamma < \omega_f$ such that $\Omega^0 = \emptyset$ and $|\, \Omega^{\alpha+1} \setminus \Omega^\alpha \,| = \aleph_{f-1}$ for all $\alpha + 1 < \gamma$ by means of the original filtration. We define $\Omega^\gamma = \bigcup_{\alpha<\gamma} \Omega^\alpha$ as usual.

Now suppose that the coordinatewise-closed $\aleph_f$-filtration $\{ \Omega^\alpha \mid \alpha \leq \gamma \}$ of Ω has been constructed up to some successor ordinal $\gamma < \omega_f$ and also satisfies $|\, \Omega^{\alpha+1} \setminus \Omega^\alpha \,| = \aleph_{f-1}$ for all $\alpha+1 \leq \gamma$. Let β be the minimal ordinal such that $\Omega^\gamma \subseteq \mho^\beta$ and $\left|\, \mho^\beta \setminus \Omega^\gamma \,\right| = \aleph_{f-1}$. Let $\Omega^{\gamma+1}_0 = \mho^\beta$ and assume we have constructed $\Omega^{\gamma+1}_n$ for some $n < \omega$. For all

$m \in [1, k]$, let

$$ {}^{\omega\uparrow}\lambda_m(\Omega_n^{\gamma+1}) = \left\{ \eta_\ell \mid \ell \in [1, k], \overline{\eta} \in \Omega_n^{\gamma+1} \cup \bigcup \Omega_n^{\gamma+1} F \right\} \cap {}^{\omega\uparrow}\lambda_m, $$

$$ \Omega_{n+1}^{\gamma+1} = {}^{\omega\uparrow}\lambda_1(\Omega_n^{\gamma+1}) \times \cdots \times {}^{\omega\uparrow}\lambda_k(\Omega_n^{\gamma+1}) \subseteq \Lambda $$

and

$$ \Omega^{\gamma+1} = \Omega \cap \bigcup_{n<\omega} \Omega_n^{\gamma+1}. $$

□

Freeness-Proposition 11. *Let $F : \Lambda \to [\Lambda_*]^{\leq \aleph_0}$ be a regressive map, $f \in [1, k]$, $\Omega \in [\Lambda]^{\aleph_{f-1}}$ and $\langle u_{\overline{\eta}} \mid \overline{\eta} \in \Omega \rangle$ be a family of subsets of $[1, k]$ such that $|u_{\overline{\eta}}| \geq f$. Then there exists a bijective enumeration $\{\overline{\eta}^\alpha \mid \alpha < \zeta\}$ of Ω for some $\zeta \in [\omega_{f-1}, \omega_f)$ such that, for all $\alpha < \zeta$, there exist $\ell_\alpha \in u_{\overline{\eta}^\alpha}$ and $n_\alpha \in [1, \omega)$ with the property that, for all $n \geq n_\alpha$,*

$$ \overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \notin \{ \overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle \mid \beta < \alpha \} \cup \bigcup \Omega_\alpha F $$

where $\Omega_\alpha = \{ \overline{\eta}^\beta \mid \beta \leq \alpha \}$.

Proof. We proceed by induction on f. If $f = 1$, then $|\Omega| = \aleph_0$ and $u_{\overline{\eta}} \neq \emptyset$ for all $\overline{\eta} \in \Omega$. For all $\alpha < \lambda_k$, define $U_\alpha = \{ \overline{\eta} \in \Omega \mid 0\eta_k = \alpha \}$. Let $N = \{ \alpha < \lambda_k \mid U_\alpha \neq \emptyset \}$ and enumerate it $N = \{ \alpha_\beta \mid \beta < \delta \}$ for some $\delta < \omega_1$ in such a way that $\alpha_\beta < \alpha_\gamma$ if and only if $\beta < \gamma < \delta$. Put $\gamma_\beta = |U_{\alpha_\beta}|$ and $\sigma_\beta = \sum_{\alpha<\beta} \gamma_\alpha$. We enumerate $U_{\alpha_\beta} = \{ \overline{\eta}^\alpha \mid \sigma_\beta \leq \alpha < \sigma_\beta + \gamma_\beta \}$. This results in a bijective enumeration $\{ \overline{\eta}^\alpha \mid \alpha < \zeta \}$ of Ω such that $\zeta \in [\omega, \omega_1)$ and, for all $\alpha < \zeta$,

$$ 0\eta_k^\alpha \leq 0\eta_k^{\alpha+1} < 0\eta_k^{\alpha+\omega}. $$

Choose $\ell_\alpha \in u_{\overline{\eta}^\alpha}$ arbitrarily. If $\overline{\eta}^\alpha \in U_\gamma$ and β_0 is the minimal ordinal such that $\overline{\eta}^{\beta_0} \in U_\gamma$, then we can find some $n_{\alpha,\beta} \in [\,1,\omega\,)$ such that $\overline{\eta}^\alpha \upharpoonleft \langle \ell_\alpha, n\rangle \neq \overline{\eta}^\beta \upharpoonleft \langle \ell_\alpha, n\rangle$ for all $\beta \in [\,\beta_0,\alpha)$ and $n \geq n_{\alpha,\beta}$. Put $n_\alpha = \max_{\beta\in[\,\beta_0,\alpha)} n_{\alpha,\beta}$. Then, for all $n \geq n_\alpha$, $\overline{\eta}^\alpha \upharpoonleft \langle \ell_\alpha, n\rangle \notin \{\, \overline{\eta}^\beta \upharpoonleft \langle \ell_\alpha, n\rangle \mid \beta \in [\,\beta_0,\alpha)\,\}$. Moreover,

$$\overline{\eta}^\alpha \upharpoonleft \langle \ell_\alpha, n\rangle \notin \{\, \overline{\eta}^\beta \upharpoonleft \langle \ell_\alpha, n\rangle \mid \beta < \beta_0 \,\} \cup \bigcup \Omega_\alpha F$$

since F is regressive and $0\eta_k^\beta < 0\eta_k^\alpha$ for all $\beta < \beta_0$.

Now suppose that the assertion is true for some $f \in [\,1,k-1\,]$. Let $\Omega \in [\Lambda]^{\aleph_f}$ and $\langle\, u_{\overline{\eta}} \mid \overline{\eta} \in \Omega \,\rangle$ with $|\, u_{\overline{\eta}}\,| \geq f+1$. Choose an $\aleph_f$-filtration $\{\, \Omega^\alpha \mid \alpha < \omega_f \,\}$ of Ω such that $\Omega^0 = \emptyset$ and $|\,\Omega^{\alpha+1} \setminus \Omega^\alpha\,| = \aleph_{f-1}$ for all $\alpha < \omega_f$. By the previous lemma, we can assume that this filtration is coordinatewise-closed. For every $\overline{\eta} \in \Omega^{\alpha+1} \setminus \Omega^\alpha$, consider

$$u^*_{\overline{\eta}} = \{\, m \in [\,1,k\,] \mid \exists \overline{\eta}' \in \Omega^\alpha,\, n < \omega\; (\,\overline{\eta} \upharpoonleft \langle m,n\rangle = \overline{\eta}' \upharpoonleft \langle m,n\rangle \;\text{ or }\; \overline{\eta} \upharpoonleft \langle m,n\rangle \in \overline{\eta}'F\,)\,\}.$$

It follows that $\left|\, u^*_{\overline{\eta}} \,\right| \leq 1$, since $\left|\, u^*_{\overline{\eta}} \,\right| > 1$ would imply that $\overline{\eta} \in \Omega^\alpha$. Put $u'_{\overline{\eta}} = u_{\overline{\eta}} \setminus u^*_{\overline{\eta}}$ and observe that $\left|\, u'_{\overline{\eta}} \,\right| \geq f$. We apply the induction hypothesis on each of the sets $\Omega^{\alpha+1} \setminus \Omega^\alpha$ together with the family $\langle\, u'_{\overline{\eta}} \mid \overline{\eta} \in \Omega^{\alpha+1} \setminus \Omega^\alpha \,\rangle$ to obtain an enumeration $\Omega^{\alpha+1} \setminus \Omega^\alpha = \langle \overline{\eta}^\beta \mid \beta < \zeta \,\rangle$ for some $\zeta \in [\omega_{f-1}, \omega_f)$ with the required property. We induce an enumeration on Ω with the desired property by ordering these enumerations lexicographically. $\square$

Now we get ready to present the $\overline{\lambda}$-Black Box, the prediction principle we will use to get rid of unwanted homomorphisms. Please keep our notation (3) from page 1 in mind.

LEMMA 12. *Let λ be an infinite cardinal and $\mathfrak{P}$ a set of cardinality $\leq \lambda^{\aleph_0}$. Then there exists a map $\Phi : {}^{\omega\uparrow}\lambda \to {}^{\omega}\mathfrak{P}$ such that, for all $\mathfrak{f} : {}^{\omega\uparrow>}\lambda \to \mathfrak{P}$ and $\nu \in {}^{\omega\uparrow>}\lambda$, we can find $\eta \in {}^{\omega\uparrow}\lambda$ with $\nu \subset \eta$ and $n\Phi_\eta = (\eta \restriction n)\mathfrak{f}$ for all $n < \omega$.*

PROOF. Since $|\,\mathfrak{P}\,| \leq \lambda^{\aleph_0} = |\,{}^{\omega}\lambda\,|$, we can fix an embedding $\pi : \mathfrak{P} \hookrightarrow {}^{\omega}\lambda$. We also fix a map $\mu : \lambda \to {}^{\omega>}\lambda$ such that the preimage $\mu^{-1}[\,\sigma\,]$ is unbounded in λ for all $\sigma \in {}^{\omega>}\lambda$.

We would like to identify every function in ${}^{n}\mathfrak{P}$ with a function in ${}^{\omega>}\lambda$ for all $n < \omega$. For this reason we define a *coding map* $\pi^n : {}^{n}\mathfrak{P} \to {}^{n^2}\lambda$ for all $n < \omega$ such that if $\varphi \in {}^{n}\mathfrak{P}$, then π^n_φ is given by $(qn + r)\pi^n_\varphi = r\pi_{q\varphi}$ for $q,r \in [\,0, n\,)$. We now consider the set

$$X = \{\, \delta \in {}^{\omega\uparrow}\lambda \mid \exists \psi_\delta \in {}^{\omega}\mathfrak{P}\ \exists i_\delta < \omega\ \forall n \geq i_\delta\ (\pi^n_{\psi_\delta \restriction n} = \mu_{n\delta})\,\}.$$

Because π is an embedding, we obtain that if $\delta \in X$, then ψ_δ is unique. We use X to define Φ in the following way: if $\delta \notin X$, we take an arbitrary $\Phi_\delta \in {}^{\omega}\mathfrak{P}$, and if $\delta \in X$, we define $\Phi_\delta = \psi_\delta$.

Let $\mathfrak{f} : {}^{\omega\uparrow>}\lambda \to \mathfrak{P}$ and $\rho \in {}^{\omega\uparrow>}\lambda$. Since we need to show that there is an extension $\eta \in {}^{\omega\uparrow}\lambda$ of ρ, we define $n\eta = n\rho$ for all $n \in \operatorname{dom}\rho$. Now assume we have defined $\eta \restriction n$ up to a certain $n \geq \operatorname{dom}\rho$. Consider the element $\psi_{\eta n} \in {}^{n}\mathfrak{P}$ given by $m\psi_{\eta n} = (\eta \restriction m)\mathfrak{f}$ for all $m < n$. Then $\pi^n_{\psi_{\eta n}} \in {}^{\omega>}\lambda$ and $\mu^{-1}\left[\pi^n_{\psi_{\eta n}}\right]$ is unbounded in λ. Define $n\eta = \alpha$ to be the least ordinal $\alpha > (n-1)\eta$ such that $\mu_\alpha = \pi^n_{\psi_{\eta n}}$. This finishes the construction of the extension $\eta \in {}^{\omega\uparrow}\lambda$ of ρ. Moreover, let $\psi_\eta = \bigcup_{n \in [\,\operatorname{dom}\rho, \omega\,)} \psi_{\eta n}$. Since $\eta \in X$ is witnessed by ψ_η and $i_\eta = \operatorname{dom}\rho$, it immediately follows that $n\Phi_\eta = n\psi_\eta = (\eta \restriction n)\mathfrak{f}$ for all $n < \omega$. $\square$

Definition 13. A finite sequence of cardinals $\overline{\lambda} = \langle \lambda_1, \ldots, \lambda_k \rangle$ is a $\blacksquare$*-sequence* (*Black Box sequence*) if for every $m \in [1, k]$, the cardinals $\chi_m = \lambda_m^{\aleph_0}$ satisfy

$$\chi_{m+1}^{\chi_m} = \chi_{m+1}.$$

Definition 14. Let $\overline{C} = \langle C_1, \ldots, C_k \rangle$ be a sequence of sets such that $|C_m| \leq \chi_m$ and take $C = \bigcup_{m \in [1,k]} C_m$. A *set-trap for* $\overline{\eta} \in \Lambda$ and $\overline{C}$ is a function $\varphi_{\overline{\eta}} : [\overline{\eta}] \to C$.

The $\overline{\lambda}$-Black Box 15. *Let* $\overline{\lambda} = \langle \lambda_1, \ldots, \lambda_k \rangle$ *be any* $\blacksquare$*-sequence,* Λ *and* Λ_* *as before,* $\overline{C} = \langle C_1, \ldots, C_k \rangle$ *and* C *as in Definition 14. Then there exists a family of set-traps* $\langle \varphi_{\overline{\eta}} \mid \overline{\eta} \in \Lambda \rangle$ *satisfying the following*

Prediction Principle*: If* $\varphi : \Lambda_* \to C$ *is any map with the trap condition* $\Lambda_m \varphi \subseteq C_m$ *and* $\alpha < \lambda_k$*, then there exists* $\overline{\eta} \in \Lambda$ *such that* $\varphi \restriction [\overline{\eta}] = \varphi_{\overline{\eta}}$ *and* $0\eta_k = \alpha$*.*

Proof. We will proceed by induction on the length of $\overline{\lambda}$.

Assume that the length of $\overline{\lambda}$ is 1, so we can simply write $\overline{\lambda} = \langle \lambda \rangle$ and $\overline{C} = \langle C \rangle$. Then $\Lambda = {}^{\omega\uparrow}\lambda$ and $\Lambda_* = {}^{\omega\uparrow>}\lambda$. Since $|C| \leq \chi = \lambda^{\aleph_0}$, Lemma 12 provides us with a map $\Phi : \Lambda \to {}^{\omega}C$. For all $\eta \in \Lambda$ and $n < \omega$, define $(\eta \restriction n)\varphi_\eta = n\Phi_\eta$ to obtain a family of set-traps $\langle \varphi_\eta \mid \eta \in \Lambda \rangle$. Now let $\varphi : \Lambda_* \to C$ be a map (here the condition $\Lambda_* \varphi \subseteq C$ is redundant), and suppose $\alpha < \lambda$. Choose an arbitrary $\nu \in \Lambda_*$ such that $0\nu = \alpha$. By Lemma 12, there exists $\eta \in \Lambda$ such that $\nu \subset \eta$ and $n\Phi_\eta = (\eta \restriction n)\varphi$ for all $n < \omega$. This means $\varphi \restriction [\eta] = \varphi_\eta$ and $0\eta = \alpha$.

Now assume that the assertion is true for some $f \in [1, k-1]$ and that the length of $\overline{\lambda}$ is $f + 1$. In this case, $\overline{\lambda} = \langle \lambda_1, \ldots, \lambda_{f+1} \rangle$ and $\overline{C} = \langle C_1, \ldots, C_{f+1} \rangle$. We also write $C^m = \bigcup_{i=1}^m C_i$ for $m \in [\,1, f+1\,]$. Define $\mathfrak{P} = {}^{\Lambda^{[1,f]}}C_{f+1}$ (the set of all maps from $\Lambda^{[1,f]}$ to C_{f+1}) and notice that $|\,\mathfrak{P}\,| \leq \chi_{f+1}^{\chi_f} = \chi_{f+1} = \lambda_{f+1}^{\aleph_0}$. Hence, Lemma 12 provides us with a map $\Phi : {}^{\omega\uparrow}\lambda_{f+1} \to {}^{\omega}\mathfrak{P}$. We would like to define the set-traps $\varphi_{\overline{\eta}}^{f+1}$ for every $\overline{\eta} \in \Lambda^{[1,f+1]}$. By induction hypothesis, we already have a family of set-traps $\langle\, \varphi_{\overline{\eta}}^{f} \mid \overline{\eta} \in \Lambda^{[1,f]}\,\rangle$. Given an $\overline{\eta} = (\eta_1, \ldots, \eta_{f+1}) \in \Lambda^{[1,f+1]}$, put $\overline{\eta}' = (\eta_1, \ldots, \eta_f) \in \Lambda^{[1,f]}$. Then for every $\overline{\eta} \in \Lambda^{[1,f+1]}$ define

$$(\overline{\eta} \mathbin{\upharpoonleft} \langle m, n\rangle)\varphi_{\overline{\eta}}^{f+1} = \begin{cases} (\overline{\eta}' \mathbin{\upharpoonleft} \langle m, n\rangle)\varphi_{\overline{\eta}'}^{f}, & \text{if } m \in [\,1, f\,]\,; \\ \overline{\eta}'\Phi_{\eta_{f+1}n}, & \text{if } m = f+1. \end{cases}$$

It remains to verify the prediction principle. Let $\varphi : \Lambda_*^{[1,f+1]} \to C^{f+1}$ be such that $\Lambda_m^{[1,f+1]}\varphi \subseteq C_m$ and $\alpha < \lambda_{f+1}$. Let $\rho \in {}^{\omega\uparrow>}\lambda_{f+1}$ with $0\rho = \alpha$. Since $\Lambda_{f+1}^{[1,f+1]}\varphi \subseteq C_{f+1}$, we can define a function $\mathfrak{f}_\nu : \Lambda^{[1,f]} \to C_{f+1}$ for every $\nu \in {}^{\omega\uparrow>}\lambda_{f+1}$ by $\overline{\eta}'\mathfrak{f}_\nu = (\overline{\eta}' \wedge \nu)\varphi$. This, in turn, gives us a function $\mathfrak{f} : {}^{\omega\uparrow>}\lambda_{f+1} \to \mathfrak{P}$. By Lemma 12, there exists $\eta \in {}^{\omega\uparrow}\lambda_{f+1}$ such that $\rho \subset \eta$ and $\Phi_{\eta n} = \mathfrak{f}_{\eta\restriction n}$ for all $n < \omega$.

Now we will use φ and η to define a function $\varphi' : \Lambda_*^{[1,f]} \to C^f$. For every $\overline{\nu} \in \Lambda_*^{[1,f]}$, define $\overline{\nu}\varphi' = (\overline{\nu} \wedge \eta)\varphi$; and observe that $\Lambda_m^{[1,f]}\varphi' \subseteq C_m$ for all $m \in [\,1, f\,]$. By induction hypothesis, there exists $\overline{\eta}' \in \Lambda^{[1,f]}$ such that $\varphi' \restriction [\,\overline{\eta}'\,] = \varphi_{\overline{\eta}'}^{f}$. Define $\overline{\eta} = \overline{\eta}' \wedge \eta$ to obtain $\eta_{f+1} = \eta$. Finally, we must verify that $\varphi \restriction [\,\overline{\eta}\,] = \varphi_{\overline{\eta}}^{f+1}$. If $m \in [\,1, f\,]$, then

$$(\overline{\eta} \mathbin{\upharpoonleft} \langle m, n\rangle)\varphi = (\overline{\eta}' \mathbin{\upharpoonleft} \langle m, n\rangle)\varphi' = (\overline{\eta}' \mathbin{\upharpoonleft} \langle m, n\rangle)\varphi_{\overline{\eta}'}^{f} = (\overline{\eta} \mathbin{\upharpoonleft} \langle m, n\rangle)\varphi_{\overline{\eta}}^{f+1},$$

and if $m = f+1$, then

$$(\overline{\eta} \mathbin{\upharpoonleft} \langle m, n\rangle)\varphi = \overline{\eta}'\mathfrak{f}_{\eta_{f+1}\restriction n} = \overline{\eta}'\Phi_{\eta_{f+1}n} = (\overline{\eta} \mathbin{\upharpoonleft} \langle m, n\rangle)\varphi_{\overline{\eta}}^{f+1}.$$

This completes the proof. $\square$

2. The Construction

Let $\overline{\lambda} = \langle \lambda_1, \ldots, \lambda_k \rangle$ be a $\blacksquare$-sequence. We consider the sets Λ and Λ_* as before. Given a subset $Y_* \subseteq \Lambda_*$, we consider the free abelian group

$$B_{Y_*} = \bigoplus_{\overline{\nu} \in Y_*} \mathbb{Z} e_{\overline{\nu}}.$$

If $Y_* = [\,\overline{\eta}\,]$ for some $\overline{\eta} \in \Lambda$, then we simply write $B_{\overline{\eta}}$. The starting point of the final construction of this section will be the abelian group

$$B = \bigoplus_{\overline{\nu} \in \Lambda_*} \mathbb{Z} e_{\overline{\nu}}.$$

Moreover, let

$$\overline{B} = \widehat{B} \cap \prod_{\overline{\nu} \in \Lambda_*} \mathbb{Z} e_{\overline{\nu}}$$

where $\widehat{B}$ denotes the p-completion of B for some prime p. Finally, let $S = \bigoplus_{n<\omega} \mathbb{Z} e_n$.

Definition 16. For every $b = \sum_{\overline{\nu} \in \Lambda_*} b_{\overline{\nu}} e_{\overline{\nu}} \in \widehat{B}$ with $b_{\overline{\nu}} \in J_p$, the Λ_*-*support* of b is the set

$$[\,b\,] = \{\,\overline{\nu} \mid b_{\overline{\nu}} \neq 0\,\}.$$

Definition 17. Let N be an abelian group. A *trap for B, N* is a homomorphism $\varphi_{\overline{\eta}} : B_{\overline{\eta}} \to N$.

THE $\overline{\lambda}$-BLACK BOX 18. *Let* $\overline{\lambda} = \langle \lambda_1, \dots, \lambda_k \rangle$ *be a* $\blacksquare$*-sequence,* Λ *and* Λ_* *as before and* N *an abelian group such that* $|N| \leq \chi_1$*. Then there exists a family of traps* $\langle \varphi_{\overline{\eta}} \mid \overline{\eta} \in \Lambda \rangle$ *satisfying the following*

Prediction Principle*: If* $\varphi : B \to N$ *is any homomorphism and* $\alpha < \lambda_k$*, then there exists* $\overline{\eta} \in \Lambda$ *such that* $\varphi \restriction B_{\overline{\eta}} = \varphi_{\overline{\eta}}$ *and* $0\eta_k = \alpha$*.*

PROOF. The members of the sequence $\overline{C} = \langle C_1, \dots, C_k \rangle$ where $C_m = N$ for all $m \in [1, k]$ satisfy $|C_m| \leq \chi_m$ since $\overline{\lambda}$ is increasing and $|N| \leq \chi_1$. Hence, $C = N$. The $\overline{\lambda}$-Black Box 15 provides us with a family of set-traps $\langle \varphi_{\overline{\eta}} \mid \overline{\eta} \in \Lambda \rangle$. Since $[\overline{\eta}] \subset \Lambda_*$, each $\varphi_{\overline{\eta}}$ can be regarded as a homomorphism $\varphi_{\overline{\eta}} : B_{\overline{\eta}} \to N$. Since any homomorphism $\varphi : B \to N$ is completely determined by its action on the generators $e_{\overline{\nu}}$ of B, it can be regarded as a function $\varphi : \Lambda_* \to C$ which also satisfies $\Lambda_m \varphi \subseteq C_m$. Thus, for $\alpha < \lambda_k$, the $\overline{\lambda}$-Black Box 15 yields that there exists $\overline{\eta} \in \Lambda$ such that $0\eta_k = \alpha$ and $\varphi \restriction [\overline{\eta}] = \varphi_{\overline{\eta}}$, i.e. $\varphi \restriction B_{\overline{\eta}} = \varphi_{\overline{\eta}}$. $\square$

DEFINITION 19. For $\overline{\eta} \in \Lambda$ and $n < \omega$, we define the *branch element* associated with $\overline{\eta}$ and n as

$$y_{\overline{\eta}n} = \sum_{i=n}^{\infty} p^{i-n} \left(\sum_{m=1}^{k} e_{\overline{\eta} \restriction \langle m,i \rangle} \right).$$

We write $y_{\overline{\eta}}$ for $y_{\overline{\eta}0}$. Choose an element $b_{\overline{\eta}} = \sum_{i=0}^{\infty} p^i b_i^{\overline{\eta}} \in \overline{B}$, where $b_i^{\overline{\eta}} \in B$ for all $i < \omega$, and put $b_{\overline{\eta}n} = \sum_{i=n}^{\infty} p^{i-n} b_i^{\overline{\eta}} \in \overline{B}$ (so $b_{\overline{\eta}} = b_{\overline{\eta}0}$). We define the *branch-like element* associated with $\overline{\eta}$ and n as

$$y'_{\overline{\eta}n} = b_{\overline{\eta}n} + y_{\overline{\eta}n}.$$

We also write $y'_{\overline{\eta}}$ for $y'_{\overline{\eta}0}$.

Since $|S| = \aleph_0 < \chi_1$, the $\overline{\lambda}$-Black Box 18 ensures the existence of a family of traps $\langle \varphi_{\overline{\eta}} \mid \overline{\eta} \in \Lambda \rangle$ such that, if $\varphi : B \to S$ is any homomorphism and $\alpha < \lambda_k$, then there exists $\overline{\eta} \in \Lambda$ with $\varphi \restriction B_{\overline{\eta}} = \varphi_{\overline{\eta}}$ and $0\eta_k = \alpha$.

For $b \in \overline{B}$, define $\| b \| = \sup_{\overline{\nu} \in [b]} \| \overline{\nu} \|$. For $\alpha < \lambda_k$, let $\overline{B}_\alpha = \langle b \in \overline{B} \mid \| b \| < \alpha \rangle$. We fix a map $\delta : \lambda_k \to \overline{B}$ such that $\alpha\delta \in \overline{B}_\alpha$ for all $\alpha < \lambda_k$.

Step Lemma 20. *Let $\overline{\eta} \in \Lambda$, $b_{\overline{\eta}} = 0\eta_k\delta$ and $\varphi_{\overline{\eta}}$ from the $\overline{\lambda}$-Black Box 18. There exists an $\varepsilon_{\overline{\eta}} \in \{0, 1\}$ such that no homomorphism $\varphi : \left\langle B, y'_{\overline{\eta}} = \varepsilon_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \right\rangle_* \to S$ satisfies both $\varphi \restriction B_{\overline{\eta}} = \varphi_{\overline{\eta}}$ and $b_{\overline{\eta}}\varphi \in \widehat{S} \setminus S$.*

Proof. Suppose towards a contradiction that for both $\varepsilon \in \{0,1\}$, there exists some $\varphi^\varepsilon : \langle B, \varepsilon b_{\overline{\eta}} + y_{\overline{\eta}} \rangle_* \to S$ such that $\varphi^\varepsilon \restriction B_{\overline{\eta}} = \varphi_{\overline{\eta}}$ and $b_{\overline{\eta}}\varphi^\varepsilon \in \widehat{S} \setminus S$. On one hand, $(b_{\overline{\eta}} + y_{\overline{\eta}})\varphi^1 - y_{\overline{\eta}}\varphi^0 \in S$. On the other hand, $(b_{\overline{\eta}} + y_{\overline{\eta}})\varphi^1 - y_{\overline{\eta}}\varphi^0 = b_{\overline{\eta}}\varphi^1 \in \widehat{S} \setminus S$, which is the desired contradiction. □

Let $\mathfrak{F} = \{ y'_{\overline{\eta}} = \varepsilon_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in \Lambda, b_{\overline{\eta}} = 0\eta_k\delta \}$ be the family of branch-like elements obtained after choosing every $\varepsilon_{\overline{\eta}}$ by means of the Step Lemma 20. Define

$$G = \langle B, y'_{\overline{\eta}n} \mid y'_{\overline{\eta}} \in \mathfrak{F}, n < \omega \rangle = \langle B, y'_{\overline{\eta}} \mid y'_{\overline{\eta}} \in \mathfrak{F} \rangle_*.$$

Notice the following divisibility property of the branch-like elements:

$$p y'_{\overline{\eta}(n+1)} = y'_{\overline{\eta}n} - \sum_{m=1}^{k} e_{\overline{\eta}\restriction\langle m,n\rangle} - \varepsilon_{\overline{\eta}} b_n^{\overline{\eta}} \in G.$$

Since $G \subseteq_* \overline{B} \subseteq \prod_{\overline{\nu} \in \Lambda_*} \mathbb{Z} e_{\overline{\nu}}$, the group G is separable.

DEFINITION 21. For $g \in G$, define the Λ-*support* $[\, g\,]_\Lambda$ of g to be the set of elements of Λ that contribute to the representation of g. More precisely, if $p^m g = b + \sum_{\overline{\eta} \in \Lambda} n_{\overline{\eta}} y'_{\overline{\eta}}$ for some $m \geq 0$, where $b \in B$ and $n_{\overline{\eta}} \in \mathbb{Z}$ for all $\overline{\eta} \in \Lambda$, then

$$[\, g\,]_\Lambda = \{\, \overline{\eta} \in \Lambda \mid n_{\overline{\eta}} \neq 0\,\}.$$

Obviously, $[\, g\,]_\Lambda$ is finite. For $H \subseteq G$, we define $[\, H\,]_\Lambda = \bigcup_{g \in H} [\, g\,]_\Lambda$.

DEFINITION 22. A module N is κ-*free* if every subset of N of cardinality $< \kappa$ is contained in a free submodule of N.

THEOREM 23. *The group G defined as above is $\aleph_k$-free.*

PROOF. Suppose that H is a subset of G of cardinality $\aleph_{k-1}$. Let $\sigma : \overline{B} \to [\, \Lambda_*\,]^{\leq \aleph_0}$ be the "Λ_*-support" function, i.e. $b\sigma = [\, b\,]$ for all $b \in \overline{B}$. Notice that $F : \Lambda \to [\, \Lambda_*\,]^{\leq \aleph_0}$ given by $\overline{\eta} F = 0\eta_k \delta\sigma$ is regressive since $\alpha\delta \in \overline{B}_\alpha$ for all $\alpha < \lambda_k$. Let $\Omega = [\, H\,]_\Lambda$, $\Omega_* = [\, H\,] \setminus ([\, \Omega\,] \cup \bigcup_{\overline{\eta} \in \Omega} [\, \overline{\eta} F\,])$ and observe that $|\, \Omega\,| \leq \aleph_{k-1}$. Then the subgroups

$$G_\Omega = \langle\, e_{\overline{\eta} \restriction \langle m,n \rangle}, e_{\overline{\nu}}, y'_{\overline{\eta}} \mid \overline{\eta} \in \Omega, \overline{\nu} \in \overline{\eta} F, m \in [\, 1, k\,]\,, n < \omega\,\rangle_*,$$

and

$$G' = G_\Omega \oplus B_{\Omega_*}$$

satisfy $H \subseteq G'$. Our goal is to show that G' is free, for which it suffices to show that G_Ω is free. Suppose that $|\, \Omega\,| = \aleph_{k-1}$. By taking $u_{\overline{\eta}} = [\, 1, k\,]$ for all $\overline{\eta} \in \Omega$, we enumerate

$\Omega = \{\, \overline{\eta}^{\alpha} \mid \alpha < \zeta \,\}$ for some $\zeta \in [\,\omega_{k-1}, \omega_k)$ according to the Freeness Proposition 11 and find $\ell_\alpha \in u_{\overline{\eta}}$ and $n_\alpha < \omega$ such that

$$\overline{\eta}^{\alpha} \upharpoonright \langle \ell_\alpha, n \rangle \notin \{\, \overline{\eta}^{\beta} \upharpoonright \langle \ell_\alpha, n \rangle \mid \beta < \alpha \,\} \cup \bigcup \Omega_\alpha F$$

for all $n \geq n_\alpha$. This allows us to write

$$G_\Omega = \langle\, e_{\overline{\eta}^{\alpha} \upharpoonright \langle m,n \rangle}, e_{\overline{\nu}}, y'_{\overline{\eta}^{\alpha} n} \mid \alpha < \zeta, \overline{\nu} \in \overline{\eta}^{\alpha} F, m \in [\,1, k\,]\,, n < \omega \,\rangle.$$

Let

$$G_\alpha = \langle\, e_{\overline{\eta}^{\beta} \upharpoonright \langle m,n \rangle}, e_{\overline{\nu}}, y'_{\overline{\eta}^{\beta} n} \mid \beta < \alpha, \overline{\nu} \in \overline{\eta}^{\beta} F, m \in [\,1, k\,]\,, n < \omega) \,\rangle$$

and notice that $G_0 = 0$, $\bigcup_{\alpha < \zeta} G_\alpha = G_\Omega$ and

$$\begin{aligned} G_{\alpha+1} &= G_\alpha + \langle\, e_{\overline{\eta}^{\alpha} \upharpoonright \langle m,n \rangle}, e_{\overline{\nu}}, y'_{\overline{\eta}^{\alpha} n} \mid \overline{\nu} \in \overline{\eta}^{\alpha} F, m \in [\,1, k\,]\,, n < \omega) \,\rangle \\ &= G_\alpha + \langle\, e_{\overline{\eta}^{\alpha} \upharpoonright \langle \ell_\alpha,n \rangle} \mid n < n_\alpha) \,\rangle + \langle\, y'_{\overline{\eta}^{\alpha} n} \mid n \geq n_\alpha \,\rangle + \langle\, e_{\overline{\nu}} \mid \overline{\nu} \in \overline{\eta}^{\alpha} F \,\rangle \\ &\qquad + \langle\, e_{\overline{\eta}^{\alpha} \upharpoonright \langle m,n \rangle} \mid m \in [1,k] \setminus \{\ell_\alpha\}, n < \omega) \,\rangle \end{aligned}$$

since

$$e_{\overline{\eta}^{\alpha} \upharpoonright \langle \ell_\alpha,n \rangle} = y'_{\overline{\eta}^{\alpha} n} - p y'_{\overline{\eta}^{\alpha}(n+1)} - b_n^{\overline{\eta}^{\alpha}} - \sum_{\substack{m=1 \\ m \neq \ell_\alpha}}^{k} e_{\overline{\eta}^{\alpha} \upharpoonright \langle m,n \rangle}$$

for $n \geq n_\alpha$ and

$$y'_{\overline{\eta}^{\alpha} n} = p^{n_\alpha - n} y'_{\overline{\eta}^{\alpha} n_\alpha} + \sum_{i=0}^{n_\alpha - n - 1} p^i \left(b_{n+i}^{\overline{\eta}^{\alpha}} + \sum_{m=1}^{k} e_{\overline{\eta}^{\alpha} \upharpoonright \langle m,n+i \rangle} \right)$$

for $n < n_\alpha$. We claim that $G_{\alpha+1}/G_\alpha$ is free. To prove our claim, suppose

$$\underbrace{\sum_{n < n_\alpha} z_n e_{\overline{\eta}^{\alpha} \upharpoonright \langle \ell_\alpha,n \rangle}}_{(1)} + \underbrace{\sum_{n \geq n_\alpha} z_n y'_{\overline{\eta}^{\alpha} n}}_{(2)} + \underbrace{\sum_{n < \omega} \sum_{\substack{m=1 \\ m \neq \ell_\alpha}}^{k} z_{mn} e_{\overline{\eta}^{\alpha} \upharpoonright \langle m,n \rangle}}_{(3)} + \underbrace{\sum_{\overline{\nu} \in \overline{\eta}^{\alpha} F} z_{\overline{\nu}} e_{\overline{\nu}}}_{(4)} \in G_\alpha$$

It is immediate that the support of term (1) is disjoint from those of the other terms. Then $z_n = 0$ for all $n < n_\alpha$ with $e_{\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n\rangle} \notin G_\alpha$. By the Freeness Proposition 11, $\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n\rangle$ neither belongs to the support of the terms (3) and (4) nor to the support of G_α for all $n \geq n_\alpha$, which implies that $z_n = 0$ for all $n \geq n_\alpha$. Since $\| b_{\overline{\eta}^\alpha} \| < 0\eta_k^\alpha$, it follows that $z_{mn} = z_{\overline{\nu}} = 0$ for all $m \in [1, k] \setminus \{ \ell_\alpha \}$, $n < \omega$ and $\overline{\nu} \in \overline{\eta}^\alpha F$ such that $e_{\overline{\eta}^\alpha \upharpoonright \langle m, n\rangle}$, $e_{\overline{\nu}} \notin G_\alpha$. Therefore, $G_{\alpha+1}/G_\alpha$ is freely generated by the set

$$\{ e_{\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, r\rangle},\ y'_{\overline{\eta}^\alpha s},\ e_{\overline{\eta}^\alpha \upharpoonright \langle m, t\rangle},\ e_{\overline{\nu}} \mid$$
$$r < n_\alpha, s \geq n_\alpha, m \in [1, k] \setminus \{ \ell_\alpha \}, t < \omega, \overline{\nu} \in \overline{\eta}^\alpha F \} \setminus G_\alpha.$$

Since G_Ω is the union of the continuous chain $\{ G_\alpha \mid \alpha < \zeta \}$ such that G_0 is free and every $G_{\alpha+1}/G_\alpha$ is free, G_Ω itself is free. We conclude that G is $\aleph_k$-free. □

THEOREM 24. *For any $\blacksquare$-sequence $\langle \lambda_1, \dots, \lambda_k \rangle$ such that $\chi_k = \lambda_k$, there exists a separable $\aleph_k$-free abelian group G with no epimorphisms onto S.*

PROOF. Let G be as before. Notice that the additional condition $\chi_k = \lambda_k$ implies that we can fix the map $\delta : \lambda_k \to \overline{B}$ to be surjective. Suppose $\varphi : G \to S$ is an epimorphism. If $\overline{B}\varphi \subseteq S$, then $e_{\overline{\nu}}\varphi \neq 0$ for only finitely many $\overline{\nu} \in \Lambda_*$. Hence, φ cannot be an epimorphism. It follows that $\overline{B}\varphi \cap \widehat{S} \setminus S \neq \emptyset$. Take any $g \in \overline{B}$ such that $g\varphi = s \in \widehat{S} \setminus S$. Then there exists $\alpha < \lambda_k$ such that $\alpha\delta = g$. By the $\overline{\lambda}$-Black Box 18, there exists $\overline{\eta} \in \Lambda$ such that $\varphi \upharpoonright B_{\overline{\eta}} = \varphi_{\overline{\eta}}$ and $0\eta_k = \alpha$. But then $\psi = \varphi \upharpoonright \langle B, y'_{\overline{\eta}} \rangle_*$ satisfies both $\psi \upharpoonright B_{\overline{\eta}} = \varphi_{\overline{\eta}}$ and $b_{\overline{\eta}}\psi = b_{\overline{\eta}}\varphi = (0\eta_k\delta)\varphi = (\alpha\delta)\varphi = g\varphi = s \in \widehat{S} \setminus S$, contradicting the choice of $\varepsilon_{\overline{\eta}}$. Therefore G has no epimorphisms onto S. □

This theorem implies that every time we have a homomorphism $\varphi : G \to H$ with H a free abelian group, then $G\varphi \subseteq H$ will have finite rank.

3. Slenderness

We have constructed a separable $\aleph_k$-free group G satisfying the property stated at the end of the previous section. We now show that this group is slender. Let $P = \prod_{n<\omega} \mathbb{Z}e_n$.

Definition 25. A torsion-free group G is *slender* if for all homomorphism $\varphi : P \to G$, we have $e_n\varphi = 0$ for almost all $n \in \omega$.

The next theorem completely characterizes slender groups as those not containing certain groups as subgroups. See Fuchs [**10**] vol. 2, Theorem 95.3 for a proof.

Theorem 26 (Nunke). *A torsion-free group is slender if and only if it contains no copy of $\mathbb{Q}$, P or J_p for any prime p.*

Assume for the moment that the next Lemma is true.

Lemma 27. *There exists a non-free subgroup of P of cardinality $\aleph_1$.*

Theorem 28. *For $k > 1$, every $\aleph_k$-free group is slender.*

Proof. Let G be an $\aleph_k$-free group for some $k > 1$. Since $\mathbb{Q}$ is not free and $|\,\mathbb{Q}\,| = \aleph_0$, G cannot contain copies of $\mathbb{Q}$. Since $\mathbb{Z}(p) \subseteq J_p$ is not free, G cannot contain copies of J_p. By Lemma 27, P has a non-free subgroup of cardinality $\aleph_1$, so G does not contain a copy of P either. Thus by Theorem 26, G is slender. □

It only remains to prove Lemma 27. First we introduce some definitions. Let R be a commutative ring with 1 and $\mathbb{S} \subseteq R \setminus \{\,0\,\}$ be multiplicatively closed.

Definition 29. An R-module M is

(1) $\mathbb{S}$-*torsion-free* if $sm = 0$ implies $m = 0$ for all $s \in \mathbb{S}$, $m \in M$;

(2) $\mathbb{S}$-*reduced* if $\bigcap_{s\in\mathbb{S}} sM = \{0\}$;

(3) $\mathbb{S}$-*divisible* if $sM = M$ for all $s \in \mathbb{S}$.

(4) A submodule N of an R-module M is $\mathbb{S}$-*dense* if $(m + sM) \cap N \neq \emptyset$ for all $m \in M$, $s \in \mathbb{S}$.

Lemma 30. *A submodule N of an R-module M is $\mathbb{S}$-dense if and only if M/N is $\mathbb{S}$-divisible.*

Proof. For $m \in M$, $s \in \mathbb{S}$, we have $(m + sM) \cap N \neq \emptyset$ if and only if there exist $m' \in M$, $n \in N$ such that $m - sm' = n$. But, this holds exactly if $m + N = sm' + N$, i.e. $M/N \subseteq s(M/N)$. $\square$

Now let $R = \mathbb{Z}$, $\mathbb{S} = \mathbb{Z}^+$ (the set of positive integers) and S as in the previous section.

Definition 31. The subgroup

$$\overline{S} = \{\, y \in P \mid \forall n \in \mathbb{Z}^+ \ (\, (y + nP) \cap S \neq \emptyset \,) \,\}$$

of P is called the $\mathbb{S}$-*closure* of S, which is the largest subgroup of P containing S as an $\mathbb{S}$-dense subgroup.

Lemma 32. *Let $S \leq G \leq P$. Then $G = \overline{S}$ if and only if P/G is $\mathbb{S}$-reduced and G/S is $\mathbb{S}$-divisible.*

PROOF. ($\Rightarrow$) Let $x \in P$ such that $(x+\overline{S}) \in \bigcap_{n\in\mathbb{Z}^+} n(P/\overline{S})$. Then for all $n \in \mathbb{Z}^+$, there exists $y_n \in P$ such that $x+\overline{S} = ny_n+\overline{S}$, so $x-ny_n \in \overline{S}$. Then for all $m, n \in \mathbb{Z}^+$, $((x-ny_n)+mP)\cap S \neq \emptyset$. In particular, $((x-ny_n)+nP)\cap S = (x+nP)\cap S \neq \emptyset$, which implies that $x \in \overline{S}$. Hence, P/G is $\mathbb{S}$-reduced. Since S is a $\mathbb{S}$-dense subgroup of $\overline{S}$, it follows that $\overline{S}/S$ is $\mathbb{S}$-divisible by Lemma 30 .

($\Leftarrow$) By Lemma 30, S is $\mathbb{S}$-dense in G, so $G \leq \overline{S}$. Let $x \in \overline{S}$. For all $n \in \mathbb{Z}^+$, $(x+nP)\cap S \neq \emptyset$. Thus, there exist $y_n \in P$ such that $x-ny_n \in S \subset G$. It follows that $x+G = ny_n+G \in \bigcap_{n\in\mathbb{Z}^+} n(P/G)$, which implies that $x \in G$ because P/G is $\mathbb{S}$-reduced. Therefore, $\overline{S} \leq G$. $\square$

The next lemma follows immediately from Fuchs [**10**], Vol. 1, Theorem 23.1 on the structure of divisible groups.

LEMMA 33. *If G is torsion-free and divisible, then G is a $\mathbb{Q}$-vector space and $G \cong \bigoplus_{i\in I} \mathbb{Q}e_i$ for some I such that $|\,I\,| = \dim_{\mathbb{Q}} G$.*

THEOREM 34. *$\overline{S}/S$ is a $\mathbb{Q}$-vector space and $\operatorname{rk}\overline{S}/S = \dim_{\mathbb{Q}}\overline{S}/S = 2^{\aleph_0}$.*

PROOF. $\overline{S}/S$ is torsion-free and divisible, so by Lemma 33, $\overline{S}/S$ is a $\mathbb{Q}$-vector space. Moreover, $|\,S\,| = \aleph_0$ and $|\,\overline{S}\,| = 2^{\aleph_0}$, so $2^{\aleph_0} = |\,\overline{S}\,| = |\,S\,| \cdot |\,\overline{S}/S\,| = \aleph_0 \cdot |\,\overline{S}/S\,|$ Therefore, $\operatorname{rk}\overline{S}/S = \dim_{\mathbb{Q}}\overline{S}/S = 2^{\aleph_0}$. $\square$

PROPOSITION 35. *Let F be a free abelian group and let $H \neq 0$ be a subgroup of F such that $|\,H\,| < |\,F\,|$. Then there exist (free) C_1, C_2 such that $F = C_1 \oplus C_2$, $H \subseteq C_1$ and $|\,C_2\,| = |\,F\,|$.*

Proof. Let $B = \{ e_i \mid i \in I \}$ be a basis of F. Every element $f \in F$ can be written as an infinite sum $\sum_{i \in I} n_i e_i$ where $n_i \in \mathbb{Z}$ and $n_i = 0$ for almost all $i \in I$. For every $f \in F$, we consider the support $[\, f\,] = \{ e_i \mid n_i \neq 0 \}$ of f, which is a finite subset of B. Put $B_1 = \bigcup_{f \in H} [\, f\,]$ and $B_2 = B \setminus B_1$. Define $C_i = \langle B_i \rangle$ for $i = 1, 2$. By construction, $C_1 \cap C_2 = \{0\}$ and $H \subseteq C_1$. Since $|\, B_1 \,| = |\, H \,|$, we conclude $|\, B_2 \,| = |\, C_2 \,| = |\, F \,|$. $\square$

Proof of Lemma 27. By Theorem 34, $\overline{S}/S$ is a $\mathbb{Q}$-vector space with $\dim_{\mathbb{Q}} \overline{S}/S = 2^{\aleph_0}$. Thus there exists a subspace $V' \leq \overline{S}/S$ such that $\dim_{\mathbb{Q}} V' = \aleph_1$. There exists some $S \leq V \leq \overline{S} \leq P$ such that $V/S \cong V'$. Since $|\, S \,| = \aleph_0$, we have $|\, V \,| = \aleph_1$. We claim that V is not free. Suppose towards a contradiction that V is free. By Proposition 35, there exist some free C_1, C_2 such that $V = C_1 \oplus C_2$, $S \subseteq C_1$ and $|\, C_2 \,| = |\, V \,|$. By construction $S \cap C_2 = \{0\}$, so that $V' \cong C_1/S \oplus C_2$ is a $\mathbb{Q}$-vector space with a free direct summand C_2, a contradiction. Therefore V is not free. $\square$

CHAPTER 2

Separable $\aleph_k$-free Groups Satisfying $\operatorname{End} G = A \oplus \operatorname{Fin} G$

1. Set-theoretic Modifications

Let A be a ring with free additive group $A^+ = \bigoplus_{\alpha<\kappa} \mathbb{Z}e_\alpha$ such that

$$\overline{A} = \widehat{A} \cap \prod_{\alpha<\kappa} \mathbb{Z}e_\alpha$$

is an A-module, where $\widehat{A}$ denotes the p-completion of A for some prime p. Such rings are called *separably realizable.* It was shown at different times and independently that $\overline{A}$ is a ring for $\kappa = \aleph_0$ by Goodearl, Menal, Moncasi [**15**], Corner, Göbel [**3**] and Nielsen [**19**]. For more details, see Göbel, Trlifaj [**14**].

We recursively construct a sequence of infinite cardinals $\langle \lambda_1, \ldots, \lambda_k \rangle$ as follows:

(1) Let $\lambda_0 = |A|$.

(2) Suppose we have constructed λ_m for some $m \in [0, k)$. Choose some cardinal μ_{m+1} such that $\mu_{m+1}^{\lambda_m} = \mu_{m+1}$ and let $\lambda_{m+1} = \mu_{m+1}^+$.

Once we have the cardinal sequence $\langle \lambda_1, \ldots, \lambda_k \rangle$, we construct the sets Λ and Λ_* as before. Given a subset $Y_* \subseteq \Lambda_*$, we consider the free A-module

$$B_{Y_*} = \bigoplus_{\overline{\nu} \in Y_*} Ae_{\overline{\nu}}.$$

The basic A-module on which we base the final construction is the free A-module

$$B = \bigoplus_{\overline{\nu} \in \Lambda_*} A e_{\overline{\nu}}.$$

Moreover, let

$$\overline{B} = \widehat{B} \cap \prod_{\overline{\nu} \in \Lambda_*} \overline{A} e_{\overline{\nu}}$$

where $\widehat{B}$ denotes the p-completion of B.

Due to the existence of non-hereditary rings, it is necessary to modify the notion of κ-free modules. The following definition of κ-freeness is due to Göbel, Herden, Shelah [**11**], which is a slightly stronger version of the one in Eklof, Mekler [**7**].

Definition 36. If κ is a regular uncountable cardinal, we say that an A-module M is *κ-free* if there is a family $\mathcal{C}$ of p-pure A-submodules of M satisfying:

(1) every element of $\mathcal{C}$ is $< \kappa$-generated and free;

(2) every element of $[M]^{<\kappa}$ is contained in an element of $\mathcal{C}$;

(3) $\mathcal{C}$ is closed under unions of well-ordered chains of length $< \kappa$.

Definition 37. Let $Y_* \subseteq \Lambda_*$, $Y \subseteq \Lambda$ and $\mathfrak{F}_{Y_* Y} = \{\, y'_{\overline{\eta}} = b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Y, b_{\overline{\eta}} \in \overline{B}_{Y_*} \,\}$ be a family of branch-like elements.

(1) If $X \subseteq Y$, then $\mathfrak{F}_{Y_* X} = \{\, y'_{\overline{\eta}} \in \mathfrak{F}_{Y_* Y} \mid \overline{\eta} \in X \,\}$.

(2) We say that $\mathfrak{F}_{Y_* Y}$ is *regressive* if $\| \, b_{\overline{\eta}} \, \| < 0\eta_k$ for all $\overline{\eta} \in Y$.

(3) We say that the pair (Y_*, Y) is Λ-*closed* if for all $\overline{\eta} \in Y$ there exists some (minimal) $N_{\overline{\eta}} < \omega$ such that $[\, \overline{\eta} \,]_{N_{\overline{\eta}}} \subseteq Y_*$ (see Definition 7).

(4) If (Y_*, Y) is Λ-closed and $\mathfrak{F}_{Y_*Y}$ is a regressive family of branch-like elements, we define the A-module

$$G_{Y_*Y} = \langle\, B_{Y_*}, Ay'_{\overline{\eta}n} \mid \overline{\eta} \in Y, n \geq N_{\overline{\eta}} \,\rangle = \langle\, B_{Y_*}, Ay'_{\overline{\eta}N_{\overline{\eta}}} \mid \overline{\eta} \in Y \,\rangle_*.$$

Observation 38. If G_{Y_*Y} is defined as before, $X \subseteq Y$, $X_* \subseteq Y_*$, (X_*, X) is a Λ-closed pair, and if $b_{\overline{\eta}} \in \overline{B}_{X_*}$ for all $\overline{\eta} \in X$, then the A-module G_{X_*X} generated by $\mathfrak{F}_{X_*X} = \mathfrak{F}_{Y_*X}$ is an A-submodule of G_{Y_*Y}.

Theorem 39. *If (Y_*, Y) is Λ-closed and $\mathfrak{F}_{Y_*Y}$ is a regressive family of branch-like elements, then G_{Y_*Y} is an $\aleph_k$-free A-module.*

Proof. We will construct a family $\mathcal{C}$ of p-pure A-submodules of G_{Y_*Y} according to Definition 36. For every $H \in [\, G_{Y_*Y} \,]^{<\aleph_k}$, let $\Omega = [\, H \,]_\Lambda$ and

$$\Omega_* = \left([\, H \,] \cup \bigcup_{\overline{\eta} \in \Omega} [\, \overline{\eta} \,] \cup [\, b_{\overline{\eta}} \,] \right) \cap Y_*.$$

Then $[\, \overline{\eta} \,]_{N_{\overline{\eta}}} \subseteq \Omega_*$ for all $\overline{\eta} \in \Omega$. It follows that (Ω_*, Ω) is Λ-closed, so together with $\mathfrak{F}_{\Omega_*\Omega}$, we can define the module $G_{\Omega_*\Omega}$ which contains H. Observe that $G_{\Omega_*\Omega}$ is $< \aleph_k$-generated because $|\, \Omega \,|$, $|\, \Omega_* \,| \leq |\, H \,| \cdot \aleph_0 < \aleph_k$. Our goal is to prove that $G_{\Omega_*\Omega}$ is free. For that purpose, let $\Omega'_* = \bigcup_{\overline{\eta} \in \Omega} [\, \overline{\eta} \,]_{N_{\overline{\eta}}} \cup [\, b_{\overline{\eta} N_{\overline{\eta}}} \,] \subseteq \Omega_*$. Since $Ae_{\overline{\nu}}$ is a free direct summand of $G_{\Omega_*\Omega}$ for every $\overline{\nu} \in \Omega_* \setminus \Omega'_*$, it is enough to show that $G_{\Omega'_*\Omega}$ is free.

Suppose that $|\, \Omega \,| = \aleph_{k-1}$. Let $F : \Lambda \to [\, \Lambda_* \,]^{\leq \aleph_0}$ be any regressive map such that $\overline{\eta} F = [\, b_{\overline{\eta}} \,]$ for all $\overline{\eta} \in \Omega$. By taking $u_{\overline{\eta}} = [\, 1, k \,]$ for all $\overline{\eta} \in \Omega$, we enumerate $\Omega = \{\, \overline{\eta}^\alpha \mid \alpha < \zeta \,\}$ for some $\zeta \in [\omega_{k-1}, \omega_k)$ according to the Freeness Proposition 11 and find $\ell_\alpha \in u_{\overline{\eta}}$ and $n_\alpha < \omega$ such that $\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n\rangle \notin \{\, \overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n\rangle \mid \beta < \alpha \,\} \cup \bigcup \Omega_\alpha F$ for all $n \geq n_\alpha$. This allows us to write

$$G_{\Omega'_*\Omega} = \langle\, Ae_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}, Ae_{\overline{\nu}}, Ay'_{\overline{\eta}^\alpha n} \mid \alpha < \zeta, \overline{\nu} \in [b_{\overline{\eta}^\alpha N_{\overline{\eta}^\alpha}}] \subseteq \overline{\eta}^\alpha F, m \in [\,1,k\,]\,, n \in [\,N_{\overline{\eta}^\alpha}, \omega)\,\rangle.$$

For convenience, we also take $n_\alpha > N_{\overline{\eta}^\alpha}$. Let

$$G_\alpha = \langle\, Ae_{\overline{\eta}^\beta \upharpoonright \langle m,n\rangle}, Ae_{\overline{\nu}}, Ay'_{\overline{\eta}^\beta n} \mid \beta < \alpha, \overline{\nu} \in [b_{\overline{\eta}^\beta N_{\overline{\eta}^\beta}}] \subseteq \overline{\eta}^\beta F, m \in [\,1,k\,]\,, n \in [\,N_{\overline{\eta}^\beta}, \omega)\,\rangle$$

and notice that $G_0 = 0$, $\bigcup_{\alpha<\zeta} G_\alpha = G_{\Omega'_*\Omega}$ and

$$\begin{aligned} G_{\alpha+1} &= G_\alpha + \langle\, Ae_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}, Ae_{\overline{\nu}}, Ay'_{\overline{\eta}^\alpha n} \mid \overline{\nu} \in [b_{\overline{\eta}^\alpha N_{\overline{\eta}^\alpha}}] \subseteq \overline{\eta}^\alpha F, m \in [\,1,k\,]\,, n \in [\,N_{\overline{\eta}^\alpha}, \omega)\,\rangle \\ &= G_\alpha + \langle\, Ae_{\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha,n\rangle} \mid n \in [\,N_{\overline{\eta}^\alpha}, n_\alpha)\,\rangle + \langle\, Ay'_{\overline{\eta}^\alpha n} \mid n \geq n_\alpha \,\rangle \\ &\quad + \langle\, Ae_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle} \mid m \in [1,k] \setminus \{\ell_\alpha\}, n \in [\,N_{\overline{\eta}^\alpha}, \omega)\,\rangle + \langle\, Ae_{\overline{\nu}} \mid \overline{\nu} \in [b_{\overline{\eta}^\alpha N_{\overline{\eta}^\alpha}}] \subseteq \overline{\eta}^\alpha F \,\rangle \end{aligned}$$

since

$$e_{\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha,n\rangle} = y'_{\overline{\eta}^\alpha n} - p y'_{\overline{\eta}^\alpha (n+1)} - b_n^{\overline{\eta}^\alpha} - \sum_{\substack{m=1 \\ m\neq \ell_\alpha}}^{k} e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}$$

for $n \geq n_\alpha$ and

$$y'_{\overline{\eta}^\alpha n} = p^{n_\alpha - n} y'_{\overline{\eta}^\alpha n_\alpha} + \sum_{i=0}^{n_\alpha - n - 1} p^i \left(b_{n+i}^{\overline{\eta}^\alpha} + \sum_{m=1}^{k} e_{\overline{\eta}^\alpha \upharpoonright \langle m,n+i\rangle} \right)$$

for $n \in [\,N_{\overline{\eta}^\alpha}, n_\alpha)$. We claim that $G_{\alpha+1} = G_\alpha \oplus \bigoplus_{e\in G'_\alpha} Ae$ for some G'_α. To prove our claim, suppose

$$\underbrace{\sum_{n\in[N_{\overline{\eta}^\alpha}, n_\alpha)} a_n e_{\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha,n\rangle}}_{(1)} + \underbrace{\sum_{n\geq n_\alpha} a_n y'_{\overline{\eta}^\alpha n}}_{(2)} + \underbrace{\sum_{n\in[N_{\overline{\eta}^\alpha},\omega)} \sum_{\substack{m=1 \\ m\neq \ell_\alpha}}^{k} a_{mn} e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}}_{(3)} + \underbrace{\sum_{\overline{\nu}\in[b_{\overline{\eta}^\alpha N_{\overline{\eta}^\alpha}}]} a_{\overline{\nu}} e_{\overline{\nu}}}_{(4)} \in G_\alpha$$

It is immediate that the support of the term (1) is disjoint from those of the other terms. Then $a_n = 0$ for all $n \in [\,N_{\overline{\eta}^\alpha}, n_\alpha)$ with $e_{\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha,n\rangle} \notin G_\alpha$. By the Freeness Proposition 11,

$\overline{\eta}^{\alpha} \upharpoonright \langle \ell_{\alpha}, n \rangle$ neither belongs to the support of the terms (3) and (4) nor to the support of G_{α} for all $n \geq n_{\alpha}$, which implies that $a_n = 0$ for all $n \geq n_{\alpha}$. Since $\| b_{\overline{\eta}^{\alpha}} \| < 0\eta_k^{\alpha}$, it follows that $a_{mn} = a_{\overline{\nu}} = 0$ for all $m \in [1, k] \setminus \{ \ell_{\alpha} \}$, $n \in [N_{\overline{\eta}^{\alpha}}, \omega)$ and $\overline{\nu} \in [b_{\overline{\eta}^{\alpha} N_{\overline{\eta}^{\alpha}}}]$ such that $e_{\overline{\eta}^{\alpha} \upharpoonright \langle m, n \rangle}$, $e_{\overline{\nu}} \notin G_{\alpha}$. Therefore $G_{\alpha+1} = G_{\alpha} \oplus \bigoplus_{e \in G'_{\alpha}} Ae$ is free with

$$\begin{aligned} G'_{\alpha} = \{ & e_{\overline{\eta}^{\alpha} \upharpoonright \langle \ell_{\alpha}, r \rangle}, \, y'_{\overline{\eta}^{\alpha} s}, \, e_{\overline{\eta}^{\alpha} \upharpoonright \langle m, t \rangle}, \, e_{\overline{\nu}} \mid \\ & r \in [N_{\overline{\eta}^{\alpha}}, n_{\alpha}), s \geq n_{\alpha}, m \in [1, k] \setminus \{ \ell_{\alpha} \}, t \in [N_{\overline{\eta}^{\alpha}}, \omega), \overline{\nu} \in [b_{\overline{\eta}^{\alpha} N_{\overline{\eta}^{\alpha}}}] \} \setminus G_{\alpha}, \end{aligned}$$

which in turn implies that $G_{\Omega'_{*}\Omega}$ is free. By setting $\mathcal{C} = \{ G_{\Omega_{*}\Omega} \mid |\Omega|, |\Omega_{*}| < \aleph_k \}$, we conclude that $G_{Y_{*}Y}$ is $\aleph_k$-free. $\square$

2. Triple Modules

In this section we introduce modules defined by triples of subsets of Λ and Λ_*. These triple modules have important freeness properties.

DEFINITION 40. Let $X_* \subseteq Y_* \subseteq \Lambda_*$, $Y \subseteq \Lambda$ and $f \in [1, k]$.

(1) Define $Y_{X_*} = \{ \overline{\eta} \in Y \mid \exists n < \omega\, ([\overline{\eta}]_n \subseteq X_*) \}$. For all $\overline{\eta} \in Y$ we define $u_{\overline{\eta}}(X_*) = \{ m \in [1, k] \mid \forall n < \omega\, ([\overline{\eta} \upharpoonleft m]_n \nsubseteq X_*) \}$.

(2) We say that the triple (Y_*, Y, X_*) is *f-closed* if (Y_*, Y) is Λ-closed, $X_* \subseteq Y_*$ and, for all $\overline{\eta} \in Y$, either $|\, u_{\overline{\eta}}(X_*) \,| \geq f$ or $\overline{\eta} \in Y_{X_*}$.

(3) A regressive family $\mathfrak{F}_{Y_*Y}$ of branch-like elements is called (Y, X_*)-*suitable* if $[\, b_{\overline{\eta}} \,] \subseteq X_*$ for all $\overline{\eta} \in Y_{X_*}$.

(4) Consider the map $\rho_{Y_*X_*} : \overline{B}_{Y_*} \to \overline{B}_{Y_*}$ defined on the elements of Y_* by

$$e_{\overline{\nu}}\rho_{Y_*X_*} = \begin{cases} 0, & \text{if } \overline{\nu} \in X_* \\ e_{\overline{\nu}}, & \text{if } \overline{\nu} \in Y_* \setminus X_* \end{cases}$$

and extended by linearity. Define $G_{Y_*YX_*} = \operatorname{Im}(\rho_{Y_*X_*} \upharpoonright G_{Y_*Y})$.

OBSERVATION 41. (1) If (Y_*, Y) is Λ-closed, $\overline{\eta} \in Y$ and $X_* \subseteq Y_*$, then $u_{\overline{\eta}}(Y_*) \subseteq u_{\overline{\eta}}(X_*)$. This fact will be used very often in the rest of this section.

(2) If $Y \subseteq Z$, $X_* \subseteq Y_* \subseteq Z_*$, the pairs (Y_*, Y), (Z_*, Z) are Λ-closed and we have a regressive family $\mathfrak{F}_{Z_*Z}$, then $G_{Y_*YX_*} \subseteq G_{Z_*ZX_*}$.

(3) Moreover, if (Y_*, Y, X_*) and (Y'_*, Y', X'_*) are two f-closed triples such that $Y'_* \subseteq Y_*$, $Y' \subseteq Y$, $X'_* \subseteq X_*$ and $(Y'_* \setminus X'_*) \cap X_* = \emptyset$, then $\rho_{Y'_*X'_*} \subseteq \rho_{Y_*X_*}$.

THEOREM 42. *Let $f \in [1,k]$, (Y_*, Y, X_*) be an f-closed triple and $\mathfrak{F}_{Y_*Y}$ be regressive and (Y, X_*)-suitable. Put $X = Y_{X_*}$. Then (X_*, X) is Λ-closed and*

$$G_{Y_*YX_*} \cong G_{Y_*Y}/G_{X_*X}$$

is $\aleph_f$-free, where G_{X_*X} is generated by $\mathfrak{F}_{X_*X} = \mathfrak{F}_{Y_*X}$.

PROOF. By the definition of Y_{X_*} there is some $N_{\overline{\eta}} \leq N'_{\overline{\eta}} < \omega$ such that $[\overline{\eta}]_{N'_{\overline{\eta}}} \subseteq X_*$. Therefore (X_*, X) is Λ-closed. Furthermore, the (Y, X_*)-suitability of $\mathfrak{F}_{Y_*Y}$ ensures that $b_{\overline{\eta}} \in \overline{B}_{X_*}$ for all $\overline{\eta} \in X$. By Observation 38, $G_{X_*X} \leq G_{Y_*Y}$. We claim $G_{X_*X} = \ker \rho_{Y_*X_*}$. We have $X_*\rho_{Y_*X_*} = 0$ by definition of $\rho_{Y_*X_*}$, and if $\overline{\eta} \in X$, then $y_{\overline{\eta}N'_{\overline{\eta}}}\rho_{Y_*X_*} = 0$. Moreover, $[b_{\overline{\eta}}] \subseteq X_*$ since $\mathfrak{F}_{Y_*Y}$ is (Y, X_*)-suitable. Hence, $y'_{\overline{\eta}N'_{\overline{\eta}}}\rho_{Y_*X_*} = 0$. Therefore, $G_{X_*X} \subseteq \ker \rho_{Y_*X_*}$. If $g \in G_{Y_*Y} \cap \ker \rho_{Y_*X_*}$, then

$$p^n g = \sum_{\overline{\nu} \in Y_*} a_{\overline{\nu}} e_{\overline{\nu}} + \sum_{\overline{\eta} \in Y \setminus X} a_{\overline{\eta}} y'_{\overline{\eta}N_{\overline{\eta}}} + \sum_{\overline{\eta} \in X} a_{\overline{\eta}} y'_{\overline{\eta}N'_{\overline{\eta}}}$$

for some $n < \omega$. Suppose that $W = \{\, \overline{\eta} \in Y \setminus X \mid a_{\overline{\eta}} \neq 0 \,\}$ is nonempty; and consider $\overline{\eta} \in W$ of maximal norm. By the f-closedness of (Y_*, Y, X_*), we obtain $|\, u_{\overline{\eta}}(X_*) \,| \geq f$. Take $m \in u_{\overline{\eta}}(X_*)$ and $n' < \omega$ large enough such that

$$\overline{\eta} \upharpoonright \langle m, n' \rangle \notin \left(\{\, \overline{\nu} \mid a_{\overline{\nu}} \neq 0 \,\} \cup \bigcup_{\overline{\eta}' \in W \setminus \{\overline{\eta}\}} [\, y_{\overline{\eta}'} \,] \right) \cap X_*,$$

which are finitely many. It follows that $a_{\overline{\eta}} y'_{\overline{\eta}N_{\overline{\eta}}}\rho_{Y_*X_*} \neq 0$. Consequently, $g \notin \ker \rho_{Y_*X_*}$, which is a contradiction. Hence, $W = \emptyset$ and $a_{\overline{\eta}} = 0$ for all $\overline{\eta} \in Y \setminus X$. Similarly, it follows that $a_{\overline{\nu}} = 0$ for all $\overline{\nu} \in Y_* \setminus X_*$. Therefore, $g \in G_{X_*X}$; and our claim is proved, which implies that $G_{Y_*YX_*} \cong G_{Y_*Y}/G_{X_*X}$.

It is only left to show that $G_{Y_*YX_*}$ is $\aleph_f$-free. We proceed as in the proof of Theorem 39 and construct a family $\mathcal{C}$ of p-pure A-submodules of $G_{Y_*YX_*}$ according to Definition 36. As in the proof of Theorem 39, for every $H \in [\, G_{Y_*Y} \setminus G_{X_*X}\,]^{<\aleph_f}$, we can find $\Omega_* \subseteq Y_*$ and $\Omega \subseteq Y$ such that $|\,\Omega\,|, |\,\Omega_*\,| \leq |\,H\,| \cdot \aleph_0$ and

$$H \subseteq G_{\Omega_*\Omega} = \langle\, Ae_{\overline{\nu}}, Ay'_{\overline{\eta}N'_{\overline{\eta}}}, G_{X_*X} \mid \overline{\nu} \in \Omega_*, \overline{\eta} \in \Omega\,\rangle_* \subseteq G_{Y_*Y},$$

where, for all $\overline{\eta} \in \Omega$, we chose some $N'_{\overline{\eta}} \geq N_{\overline{\eta}}$ such that $[\,\overline{\eta}\,]_{N'_{\overline{\eta}}} \subseteq \Omega_*$. Then $G_{\Omega_*\Omega}/G_{X_*X}$ is $< \aleph_f$-generated. Let $\Omega'_* = X_* \cup \bigcup_{\overline{\eta}\in\Omega} [\,\overline{\eta}\,]_{N'_{\overline{\eta}}} \cup [b_{\overline{\eta}N'_{\overline{\eta}}}]$ and $\Delta = \Omega_* \setminus \Omega'_*$. Notice that B_Δ is a free summand of $G_{\Omega_*\Omega}$ and that $B_\Delta \rho_{Y_*X_*}$ is a free summand of $G_{\Omega_*\Omega}\rho_{Y_*X_*}$. Let

$$G_\Omega = \langle\, Ae_{\overline{\eta}\restriction\langle m,n\rangle}, Ae_{\overline{\nu}}, Ay'_{\overline{\eta}N'_{\overline{\eta}}}, G_{X_*X} \mid \overline{\eta} \in \Omega \setminus X, \overline{\nu} \in [\, b_{\overline{\eta}N'_{\overline{\eta}}}\,], m \in [\,1,k\,]\,, n \in [N'_{\overline{\eta}}, \omega)\,\rangle_*.$$

Then, $G_{\Omega'_*\Omega} \subseteq G_\Omega$ and

$$G_\Omega \rho_{Y_*X_*} \cong G_\Omega/G_{X_*X} \subseteq_* G_{Y_*Y}/G_{X_*X}.$$

Consequently, $G_\Omega \rho_{Y_*X_*}$ is pure (see Fuchs [**10**], Vol.1, Lemma 26.1(ii)).

Suppose that $|\,\Omega\,| = \aleph_{f-1}$. Let $F : \Lambda \to [\,\Lambda_*\,]^{\leq\aleph_0}$ be any regressive map such that $\overline{\eta}F = [\,b_{\overline{\eta}}\,]$ for all $\overline{\eta} \in \Omega$. By taking $u_{\overline{\eta}} = u_{\overline{\eta}}(X_*)$ for all $\overline{\eta} \in \Omega \setminus X$ (which satisfy $|\,u_{\overline{\eta}}(X_*)\,| \geq f$), we enumerate $\Omega = \{\,\overline{\eta}^\alpha \mid \alpha < \zeta\,\}$ for some $\zeta \in [\,\omega_{f-1}, \omega_f)$ according to the Freeness Proposition 11, and write

$$\begin{aligned} G_\Omega = \langle\, & Ae_{\overline{\eta}^\alpha\restriction\langle m,n\rangle}, Ae_{\overline{\nu}}, Ae_{\overline{\nu}'}, Ay'_{\overline{\eta}^\alpha n}, Ay'_{\overline{\eta}' n'} \mid \\ & \alpha < \zeta, m \in [\,1,k\,]\,, n \in [\,N'_{\overline{\eta}}, \omega\,), \overline{\nu} \in [\,b_{\overline{\eta}^\alpha N'_{\overline{\eta}^\alpha}}\,], \overline{\nu}' \in X_*, \overline{\eta}' \in X, n' \in [\,N_{\overline{\eta}'}, \omega\,)\,\rangle. \end{aligned}$$

We also find $\ell_\alpha \in u_{\overline{\eta}^\alpha}$ and $n_\alpha < \omega$ such that

$$\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \notin \{\, \overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle \mid \beta < \alpha \,\} \cup \bigcup \Omega_\alpha F$$

for all $n \geq n_\alpha$. Moreover,

$$\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \notin \{\, \overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle \mid \beta < \alpha \,\} \cup \bigcup \Omega_\alpha F \cup X_*$$

for infinitely many $n \geq n_\alpha$. Let

$$G_\alpha = \langle\, Ae_{\overline{\eta}^\beta \upharpoonright \langle m,n \rangle}, Ae_{\overline{\nu}}, Ae_{\overline{\nu}'}, Ay'_{\overline{\eta}^\beta n}, Ay'_{\overline{\eta}' n'} \mid$$

$$\beta < \alpha, m \in [\,1, k\,]\,, n \in [\,N'_{\overline{\eta}}, \omega\,), \overline{\nu} \in [\,b_{\overline{\eta}^\beta N'_{\overline{\eta}^\beta}}\,], \overline{\nu}' \in X_*, \overline{\eta}' \in X, n' \in [\,N_{\overline{\eta}'}, \omega\,)\,\rangle.$$

As in Theorem 39, it follows that $G_0 = G_{X_*X}$, $G_\zeta = G_\Omega$ and that G_α is a direct summand of $G_{\alpha+1}$, with free complement in $G_{\alpha+1}$. We conclude that $H\rho_{Y_*X_*}$ is a subset of the pure free A-submodule $(B_\Delta \oplus G_\Omega)\rho_{Y_*X_*} = B_\Delta\rho_{Y_*X_*} \oplus G_\Omega\rho_{Y_*X_*}$ of $G_{Y_*YX_*}$.

By setting $\mathcal{C} = \{\, (B_\Delta \oplus G_\Omega)\rho_{Y_*X_*} \mid |\,\Delta\,|, |\,\Omega\,| < \aleph_f \,\}$, we conclude that $G_{Y_*YX_*}$ is $\aleph_f$-free. □

Lemma 43. *Let (Z_*, Z, Y_*) and (Y_*, Y, X_*) be two f-closed triples such that $Y = Z_{Y_*}$. Then*

(1) (Z_*, Z, X_*) is f-closed and $Z_{X_*} = Y_{X_*}$.

(2) If $\mathfrak{F}_{Z_*Z}$ is a regressive (Z, Y_*)-suitable family of branch-like elements such that $\mathfrak{F}_{Z_*Y} \subseteq \mathfrak{F}_{Z_*Z}$ is (Y, X_*)-suitable, then

$$G_{Z_*ZX_*}/G_{Y_*YX_*} \cong G_{Z_*ZY_*}.$$

PROOF. (1) On the one hand, it is immediate that $Y_{X_*} \subseteq Z_{X_*}$; on the other hand, $Z_{X_*} \subseteq Z_{Y_*} = Y$, so $Z_{X_*} \subseteq Y_{X_*}$.

If $\overline{\eta} \in Z \setminus Y$, then $|\, u_{\overline{\eta}}(Y_*) \,| \geq f$, which implies that $|\, u_{\overline{\eta}}(X_*) \,| \geq f$ because of $u_{\overline{\eta}}(Y_*) \subseteq u_{\overline{\eta}}(X_*)$. If $\overline{\eta} \in Y$, then either $|\, u_{\overline{\eta}}(X_*) \,| \geq f$ or $\overline{\eta} \in Y_{X_*}$ by the f-closedness of (Y_*, Y, X_*).

(2) Observe that

$$G_{Z_*ZX_*}/G_{Y_*YX_*} \cong \left(G_{Z_*Z}/G_{X_*Z_{X_*}}\right) / \left(G_{Y_*Y}/G_{X_*Y_{X_*}}\right) \cong G_{Z_*Z}/G_{Y_*Y} \cong G_{Z_*ZY_*}. \quad \square$$

DEFINITION 44. If (Y_*, Y) is Λ-closed and if $\mathfrak{F}_{Y_*Y}$ is a regressive family of branch-like elements, then we say that $\Omega_* \subseteq Y_*$ is $\mathfrak{F}_{Y_*Y}$*-closed* if (Y_*, Y, Ω_*) is k-closed and $\mathfrak{F}_{Y_*Y}$ is (Y, Ω_*)-suitable.

DEFINITION 45. For $Y \subseteq \Lambda$ and $X_* \subseteq \Lambda_*$, define

$$Y(X_*) = \{\, \overline{\eta} \in Y \mid |\, u_{\overline{\eta}}(X_*) \,| < k \,\}.$$

THEOREM 46. *If (Y_*, Y) is Λ-closed, $X_* \subseteq Y_*$ and $\mathfrak{F}_{Y_*Y}$ is a regressive family of branch-like elements, then there exists an $\mathfrak{F}_{Y_*Y}$-closed $\Omega_* \subseteq Y_*$ such that $X_* \subseteq \Omega_*$ and $|\, \Omega_* \,| \leq |\, X_* \,|^{\aleph_0}$.*

PROOF. Let $\Omega_0 = X_*$ and $\Omega_\gamma = \bigcup_{\alpha<\gamma} \Omega_\alpha$ for all limit ordinals $0 < \gamma < \omega_1$. For $\alpha < \omega_1$, put

$$\Omega_{\alpha+1} = \Omega_\alpha \cup \bigcup_{\overline{\eta} \in Y(\Omega_\alpha)} [\,\overline{\eta}\,]_{N_{\overline{\eta}}} \cup [\, b_{\overline{\eta}} \,]\,.$$

Observe that $|\,\Omega_\alpha\,| \leq |\,X_*\,|^{\aleph_0}$ recursively, since $|\,Y(\Omega_\alpha)\,| \leq |\,\Omega_\alpha\,|^{\aleph_0} \leq |\,X_*\,|^{\aleph_0}$. By defining $\Omega_* = \bigcup_{\alpha<\omega_1} \Omega_\alpha$, we obtain $|\,\Omega_*\,| \leq |\,X_*\,|^{\aleph_0}$ as well. If $\overline{\eta} \in Y$ and $|\,u_{\overline{\eta}}(\Omega_*)\,| < k$, then $\overline{\eta} \in Y(\Omega_*)$, which in turn implies that $\overline{\eta} \in Y(\Omega_\alpha)$ for some $\alpha < \omega_1$. It follows $[\,\overline{\eta}\,]_{N_{\overline{\eta}}} \subseteq \Omega_{\alpha+1}$ and $\overline{\eta} \in Y_{\Omega_{\alpha+1}} \subseteq Y_{\Omega_*}$. Therefore, (Y_*, Y, Ω_*) is k-closed. Now, if $\overline{\eta} \in Y_{\Omega_*}$, then $\overline{\eta} \in Y(\Omega_\alpha)$ for some $\alpha < \omega_1$, which implies $[\,b_{\overline{\eta}}\,] \subseteq \Omega_{\alpha+1} \subseteq \Omega_*$. Hence, $\mathfrak{F}_{Y_*Y}$ is (Y, Ω_*)-suitable. $\square$

Observation 47. In the previous theorem, it is necessary to take the union of the sets Ω_α according to ω_1 and not according to ω, since it is possible that, for some $\overline{\eta} \in Y$, $|\,[\,\overline{\eta}\,] \cap \Omega_n\,| < \aleph_0$ for all $n < \omega$, but $|\,[\,\overline{\eta}\,] \cap \Omega_\omega\,| = \aleph_0$. Thus we cannot guarantee either that $|\,u_{\overline{\eta}}(\Omega_\omega)\,| = k$ or $\overline{\eta} \in Y_{\Omega_\omega}$.

Definition 48. If (Y_*, Y) is Λ-closed, $X_* \subseteq Y_*$ and if $\mathfrak{F}_{Y_*Y}$ is a regressive family of branch-like elements, then we say that $\Omega_* \subseteq Y_*$ is the *closure of* X_* with respect to (Y_*, Y) and $\mathfrak{F}_{Y_*Y}$ provided Ω_* is constructed as in Theorem 46. We write $\Omega_* = \overline{X}_*(Y_*, Y, \mathfrak{F}_{Y_*Y})$.

In the next lemmas and Theorem 55, we want to "cover" small submodules of a triple module by small triple modules. The "small covers" are obtained by simple closure arguments (like elementary submodels) from the given large triple module.

Lemma 49. *Let (Z_*, Z, Y_*) and (Y_*, Y, X_*) be two f-closed triples such that $Y = Z_{Y_*}$, $\mathfrak{F}_{Z_*Z}$ is regressive and (Z, Y_*)-suitable, $\mathfrak{F}_{Z_*Y} \subseteq \mathfrak{F}_{Z_*Z}$ is (Y, X_*)-suitable and $H \subseteq G_{Y_*YX_*}$. Then there exists $\Omega_* \subseteq Y_*$ such that*

(1) $|\,\Omega_*\,| \leq |\,H\,|^{\aleph_0}$,

(2) $(Z_*, Z, X_* \cup \Omega_*)$ *and* $(X_* \cup \Omega_*, Y', X_*)$ *are* f*-closed, where* $Y' = Z_{X_* \cup \Omega_*} = Y_{X_* \cup \Omega_*}$,

(3) $\mathfrak{F}_{Z_* Z}$ *is* $(Z, X_* \cup \Omega_*)$*-suitable,*

(4) $\mathfrak{F}_{Z_* Y'}$ *is* (Y', X_*)*-suitable and*

(5) $H \subseteq G_{X_* \cup \Omega_*\, Y' X_*} \subseteq G_{Y_* Y X_*} \subseteq G_{Z_* Z X_*}$.

Proof. (1) For every $g \in H$, choose some $g' \in G_{Y_* Y}$ such that $g' \rho_{Z_* X_*} = g$; and gather them in a set H'. Clearly $[\,H\,] \subseteq [\,H'\,] \setminus X_*$. Put $\Omega_* = \overline{[H']}(Y_*, Y, \mathfrak{F}_{Z_* Y})$. Then $|\,\Omega_*\,| \leq |\,H\,|^{\aleph_0}$ follows by Theorem 46.

(2) For $\overline{\eta} \in Z$, we have either $|\,u_{\overline{\eta}}(Y_*)\,| \geq f$ or $\overline{\eta} \in Z_{Y_*}$. If $|\,u_{\overline{\eta}}(Y_*)\,| \geq f$, then $|\,u_{\overline{\eta}}(X_* \cup \Omega_*)\,| \geq f$ since $X_* \cup \Omega_* \subseteq Y_*$. Otherwise, we have $\overline{\eta} \in Z_{Y_*} = Y$, so either $|\,u_{\overline{\eta}}(X_*)\,| \geq f$ or $\overline{\eta} \in Y_{X_*} \subseteq Z_{X_* \cup \Omega_*}$. If $|\,u_{\overline{\eta}}(X_*)\,| \geq f$, then either $|\,u_{\overline{\eta}}(\Omega_*)\,| = k$ or $\overline{\eta} \in Y_{\Omega_*} \subseteq Z_{X_* \cup \Omega_*}$ since (Y_*, Y, Ω_*) is k-closed. If $|\,u_{\overline{\eta}}(\Omega_*)\,| = k$, then $|\,u_{\overline{\eta}}(X_* \cup \Omega_*)\,| = |\,u_{\overline{\eta}}(X_*)\,| \geq f$. Hence, $(Z_*, Z, X_* \cup \Omega_*)$ is f-closed.

Notice that $Y_{X_* \cup \Omega_*} \subseteq Z_{X_* \cup \Omega_*}$ and that $Z_{X_* \cup \Omega_*} \subseteq Z_{Y_*} = Y$. Consequently, $Z_{X_* \cup \Omega_*} \subseteq Y_{X_* \cup \Omega_*}$. Since (Y_*, Y, X_*) is f-closed and $Y'_{X_*} = Y_{X_*}$, $(X_* \cup \Omega_*, Y', X_*)$ is also f-closed.

(3) Let $\overline{\eta} \in Y'$. If $|\,u_{\overline{\eta}}(\Omega_*)\,| = k$, then $\overline{\eta} \in Y_{X_*}$, so $[\,b_{\overline{\eta}}\,] \subseteq X_*$ because $\mathfrak{F}_{Z_* Y}$ is (Y, X_*)-suitable. If $\overline{\eta} \in Y_{\Omega_*}$, then $[\,b_{\overline{\eta}}\,] \subseteq \Omega_*$ by Theorem 46.

(4) Follows from the (Y, X_*)-suitability of $\mathfrak{F}_{Z_* Y}$ since $Y'_{X_*} = Y_{X_*}$.

(5) The second inclusion follows from Observation 41. $\square$

LEMMA 50. *Let (Z_*, Z, Y_*) and (Y_*, Y, X_*) be two f-closed triples such that $Y = Z_{Y_*}$, $\mathfrak{F}_{Z_*Z}$ is regressive and (Z, Y_*)-suitable, $\mathfrak{F}_{Z_*Y} \subseteq \mathfrak{F}_{Z_*Z}$ is (Y, X_*)-suitable and $K \subseteq G_{Z_*ZX_*}$. Then there exist $\Omega \subseteq Z$ and $\Omega_* \subseteq Z_*$ such that*

(1) $|\,\Omega\,|, |\,\Omega_*\,| \leq |\,K\,| \cdot \aleph_0$,

(2) $(Y_* \cup \Omega_*, Y \cup \Omega, Y_*)$ is f-closed and $(Y \cup \Omega)_{Y_*} = Y$,

(3) $\mathfrak{F}_{Z_*(Y\cup\Omega)}$ is $(Y \cup \Omega, Y_*)$-suitable,

(4) $K \subseteq G_{(Y_*\cup\Omega_*)\,(Y\cup\Omega)\,X_*} \subseteq G_{Z_*ZX_*}$ and

(5) If (Z_*, Z, X_*) is f'-closed, then $(Y_* \cup \Omega_*, Y \cup \Omega, X_*)$ is also f'-closed.

PROOF. (1) Let $Z'_* = Z_* \setminus X_*$ and $Z' = Z \setminus Z_{X_*}$. For every $g \in K$, choose some $g' \in G_{Z'_*Z'}$ such that $g'\rho_{Z_*X_*} = g$; and gather them in a set K'. Clearly $[\,K\,] \subseteq [\,K'\,] \setminus X_*$. Let $\Omega_* = ([\,K'\,] \cup [\,[\,K'\,]_\Lambda\,]) \cap Z_* \subseteq Z'_*$ and $\Omega = Z_{\Omega_*}$.

(2) By the Λ-closedness of (Y_*, Y), it follows that $Y = Y_{Y_*} \subseteq (Y \cup \Omega)_{Y_*} \subseteq Z_{Y_*} = Y$. Moreover, $(Y_* \cup \Omega_*, Y \cup \Omega, Y_*)$ is f-closed, since (Z_*, Z, Y_*) is f-closed.

(3) Is immediate by the (Z, Y_*)-suitability of $\mathfrak{F}_{Z_*Z}$ and the fact that $(Y \cup \Omega)_{Y_*} = Z_{Y_*}$.

(4) The second inclusion follows from Observation 41.

(5) Assume (Z_*, Z, X_*) is f'-closed; and let $\overline{\eta} \in Y \cup \Omega$. Then either $|\,u_{\overline{\eta}}(X_*)\,| \geq f'$ or $\overline{\eta} \in Z_{X_*} = Y_{X_*} \subseteq (Y \cup \Omega)_{X_*}$. $\square$

THEOREM 51. *Let (Z_*, Z, Y_*) and (Y_*, Y, X_*) be two f-closed triples such that $Y = Z_{Y_*}$, $\mathfrak{F}_{Z_*Z}$ is regressive and (Z, Y_*)-suitable, $\mathfrak{F}_{Z_*Y} \subseteq \mathfrak{F}_{Z_*Z}$ is (Y, X_*)-suitable, $H \subseteq G_{Y_*YX_*}$ and $K \subseteq G_{Z_*ZX_*}$ such that $|H|, |K| \leq \kappa$. Then there exist Z'_*, Y'_*, $X'_* \subseteq \Lambda_*$, Z', $Y' \subseteq \Lambda$ such that*

(1) $Z'_* \subseteq Z_*$, $Y'_* \subseteq Y_*$, $X'_* \subseteq X_*$, $Z' \subseteq Z \setminus Z_{X_*}$, $Y' = Z'_{Y'_*} = Z_{Y'_*} \setminus Z_{X_*}$,

(2) $Z'_{X_*} = Y'_{X_*} = \emptyset$ *and* $|Z'_*|, |Z'|, |Y'_*|, |Y'|, |X'_*| \leq \kappa^{\aleph_0}$,

(3) *the triples* (Z'_*, Z', Y'_*), (Y'_*, Y', X'_*) *are f-closed,*

(4) $\mathfrak{F}_{Z_*Z'} \subseteq \mathfrak{F}_{Z_*Z}$ *is* (Z', Y'_*)*-suitable and* $\mathfrak{F}_{Z_*Y'} \subseteq \mathfrak{F}_{Z_*Y}$ *is* (Y', X'_*)*-suitable,*

(5) $H \subseteq G_{Y'_*Y'X'_*} \subseteq G_{Y_*YX_*}$ *and* $K \subseteq G_{Z'_*Z'X'_*} \subseteq G_{Z_*ZX_*}$,

(6) *If* (Z_*, Z, X_*) *is f'-closed, then* (Z'_*, Z', X'_*) *is also f'-closed.*

PROOF. Apply Lemma 49 to H to obtain Ω^1_*; and let $Y_1 = Z_{X_* \cup \Omega^1_*} = Y_{X_* \cup \Omega^1_*}$. Apply Lemma 50 to K to obtain Ω^2 and Ω^2_*. Define $Z'_* = \Omega^1_* \cup \Omega^2_*$, $Y'_* = \Omega^1_*$, $X'_* = X_* \cap \Omega^1_*$, $Z' = (Y_1 \cup \Omega^2) \setminus (Y_1 \cup \Omega^2)_{X_*}$. Then $Z'_{Y'_*} = (Y_1 \cup \Omega^2)_{\Omega^1_*} \setminus (Y_1 \cup \Omega^2)_{X_*} = Z_{Y'_*} \setminus Z_{X_*}$. Observe that $\Omega^2 \setminus Z_{X_*} \subseteq Z'$ (which justifies condition (f) from Theorem 55).

(1) and (2) are immediate by definition.

(3) Let $\overline{\eta} \in Z'$. If $\overline{\eta} \in Y_1$, then $\overline{\eta} \in Z'_{Y'_*} \subseteq Z'_{Z'_*}$. Otherwise, $\overline{\eta} \in \Omega^2$, and $\overline{\eta} \in Z'_{\Omega^1_* \cup \Omega^2_*} = Z'_{Z'_*}$. Therefore, (Z'_*, Z') is Λ-closed. If $|u_{\overline{\eta}}(X_* \cup Y'_*)| \geq f$, then $|u_{\overline{\eta}}(Y'_*)| \geq f$. Otherwise, $|u_{\overline{\eta}}(X_* \cup Y'_*)| < f$, and $\overline{\eta} \in (Y_1 \cup \Omega^2)_{X_* \cup Y'_*} \cap Z' = Z'_{Y'_*}$. Therefore, (Z'_*, Z', Y'_*) is f-closed.

By definition of Y', the pair (Y'_*, Y') is Λ-closed. Let $\overline{\eta} \in Y' \subseteq Y_1$. It is not possible to have $|u_{\overline{\eta}}(X'_*)| < f$, since it would imply $|u_{\overline{\eta}}(X_*)| < f$, so

$\overline{\eta} \in (Y_1)_{X_*} \cap Y' = \emptyset$. Hence, $|\, u_{\overline{\eta}}(X'_*)\,| \geq f$ for all $\overline{\eta} \in Y'$ and (Y'_*, Y', X'_*) is f-closed.

(4) If $\overline{\eta} \in Z'_{Y'_*} = Y'$, then $[\, b_{\overline{\eta}}\,] \subseteq Y'_* = \Omega^1_*$ by construction of Ω^1_*, so $\mathfrak{F}_{Z_*Z'} \subseteq \mathfrak{F}_{Z_*Z}$ is (Z', Y'_*)-suitable. Since $Y'_{X'_*} = \emptyset$, $\mathfrak{F}_{Z_*Y'} \subseteq \mathfrak{F}_{Z_*Y}$ is (Y', X'_*)-suitable.

(5) Follows by Observation 41.

(6) Let $\overline{\eta} \in Z'$ and suppose $|\, u_{\overline{\eta}}(X'_*)\,| < f'$. It follows $|\, u_{\overline{\eta}}(X_*)\,| < f'$, so $\overline{\eta} \in Z_{X_*} \cap Z' = \emptyset$, which is a contradiction. Hence, $|\, u_{\overline{\eta}}(X'_*)\,| \geq f'$ for all $\overline{\eta} \in Z'$; and (Z'_*, Z', X'_*) is f'-closed. □

Observation 52. It is worth to remark that the construction of the triple (Y'_*, Y', X'_*) obtained in the last theorem depends only on (Y_*, Y, X_*) and H, and not on (Z_*, Z).

Definition 53. Let $\upsilon = (\overline{\nu}_1, \overline{\nu}_2) \in \Lambda_* \times \Lambda_*$ and $X_* \subseteq Y_* \subseteq \Lambda_*$.

(1) $\overline{\nu}_1$ and $\overline{\nu}_2$ are *comparable* if there exists a unique $\overline{\eta}^\upsilon \in \Lambda$ such that $\overline{\nu}_1, \overline{\nu}_2 \in [\,\overline{\eta}^\upsilon\,]$.

(2) Define $\mathrm{Cp}(X_*) = \{\, \upsilon = (\overline{\nu}_1, \overline{\nu}_2) \mid \overline{\nu}_1, \overline{\nu}_2 \in X_* \text{ are comparable}\,\} \subseteq X_* \times X_*$.

(3) We say that X_* is *pairwise closed for* Y_* if $[\,\overline{\eta}^\upsilon\,] \cap Y_* \subseteq X_*$ for all $\upsilon \in \mathrm{Cp}(X_*)$.

Lemma 54. *If* $X_* \subseteq Y_* \subseteq \Lambda_*$*, then there is a minimal pairwise closed set* $\mathrm{PC}(X_*, Y_*)$ *such that* $X_* \subseteq \mathrm{PC}(X_*, Y_*) \subseteq Y_*$ *and* $|\,\mathrm{PC}(X_*, Y_*)\,| \leq |\, X_*\,| \cdot \aleph_0$.

Proof. Let $X^0_* = X_*$. For $n < \omega$, let

$$X^{n+1}_* = X^n_* \cup \bigcup_{\upsilon \in \mathrm{Cp}(X^n_*)} (\,[\,\overline{\eta}^\upsilon\,] \cap Y_*\,).$$

Put

$$\mathrm{PC}(X_*, Y_*) = \bigcup_{n\in\omega} X_*^n.$$

Since $|\,\mathrm{Cp}(X_*)\,| \le |\,X_*\,|$, we have $|\,\mathrm{PC}(X_*, Y_*)\,| \le |\,X_*\,| \cdot \aleph_0$. $\square$

THEOREM 55. *Let $f \in [2, k]$, (Z_*, Z, Y_*) and (Y_*, Y, X_*) be two f-closed triples such that $Y = Z_{Y_*}$ and $\mathfrak{F}_{Z_*Z}$ be regressive and (Z, Y_*)-suitable such that $\mathfrak{F}_{Z_*Y} \subseteq \mathfrak{F}_{Z_*Z}$ is (Y, X_*)-suitable. Let $\Omega_*^1 \subseteq Y_*$, Ω_*^2, $\Omega_*^{2,n} \subseteq Z_*$ $(n < \omega)$, Z', $Z_n \subseteq Z$ $(n < \omega)$ be such that*

(a) Ω_*^1 *is* $\mathfrak{F}_{Z_*Y}$*-closed,* Ω_*^2, $\Omega_*^{2,n}$ *are* $\mathfrak{F}_{Z_*Z}$*-closed,*

(b) $\mathrm{PC}(\Omega_*^{2,n}, Z_*) \subseteq \Omega_*^{2,n+1}$,

(c) $(\Omega_*^1 \cup \Omega_*^2, Z', \Omega_*^1)$, $(\Omega_*^1 \cup \Omega_*^{2,n}, Z_n, \Omega_*^1)$ $(n < \omega)$ *and* $(\Omega_*^1, Y', X_* \cap \Omega_*^1)$ *are f-closed with* $Y' = Z'_{\Omega_*^1} = (Z_n)_{\Omega_*^1}$ $(n < \omega)$,

(d) $\mathfrak{F}_{Z_*Z'}$ *is* (Z', Ω_*^1)*-suitable,* $\mathfrak{F}_{Z_*Z_n}$ *is* (Z_n, Ω_*^1)*-suitable* $(n < \omega)$ *and* $\mathfrak{F}_{Z_*Y'}$ *is* $(Y', X_* \cap \Omega_*^1)$*-suitable,*

(e) $Z'_{X_*} = Y'_{X_*} = (Z_n)_{X_*} = \emptyset$ *and*

(f) $Z_{\Omega_*^2} \setminus Z_{X_*} \subseteq Z'$ *and* $Z_{\Omega_*^{2,n}} \setminus Z_{X_*} \subseteq Z_n$.

If we let

(i) $Z''_* = \Omega_*^1 \cup \Omega_*^2 \cup \bigcup_{n<\omega} \Omega_*^{2,n}$,

(ii) $Y''_* = \Omega_*^1 \cup \bigcup_{n<\omega} \Omega_*^{2,n}$,

(iii) $X''_* = X_* \cap \Omega_*^1$,

(iv) $Z'' = Z' \cup \bigcup_{n<\omega} Z_n$ *and*

(v) $Y'' = Y' \cup \bigcup_{n<\omega} Z_n$, *then the following holds:*

(1) $Z'' \subseteq Z \setminus Z_{X_*}$ *and* $Z''_{X_*} = Y''_{X_*} = \emptyset$,

(2) (Z''_*, Z'', Y''_*) *is* $(f-1)$*-closed,* (Y''_*, Y'', X''_*) *is* f*-closed and* $Y'' = Z''_{Y''_*}$,

(3) $\mathfrak{F}_{Z_* Z''}$ *is* (Z'', Y''_*)*-suitable and* $\mathfrak{F}_{Z_* Y''}$ *is* (Y'', X''_*)*-suitable,*

(4) $G_{Y''_* Y'' X''_*} = G_{\Omega^1_* \, Y' \, X_* \cap \Omega^1_*} + \sum_{n<\omega} G_{(\Omega^1_* \cup \Omega^{2,n}_*) \, Z_n \, X_* \cap \Omega^1_*}$ *and*

$G_{Z''_* Z'' X''_*} = G_{(\Omega^1_* \cup \Omega^2_*) \, Z' \, X_* \cap \Omega^1_*} + \sum_{n<\omega} G_{(\Omega^1_* \cup \Omega^{2,n}_*) \, Z_n \, X_* \cap \Omega^1_*}$*, and*

(5) *if* (Z_*, Z, X_*) *is* f'*-closed then* (Z''_*, Z'', X''_*) *is also* f'*-closed.*

Proof. (1) It is immediate that Z', $Z_n \subseteq Z \setminus Z_{X_*}$ by (e). The rest is clear.

(2) (Z''_*, Z'') and (Y''_*, Y'') are Λ-closed since $(\Omega^1_* \cup \Omega^2_*, Z')$, (Ω^1_*, Y') and all pairs $(\Omega^1_* \cup \Omega^{2,n}_*, Z_n)$ are Λ-closed.

Let $\overline{\eta} \in Z''$. If $|\, u_{\overline{\eta}}(\Omega^1_*) \,| < f$, then $\overline{\eta} \in Z''_{\Omega^1_*} \subseteq Z''_{Y''_*}$ since $(\Omega^1_* \cup \Omega^2_*, Z', \Omega^1_*)$ and all triples $(\Omega^1_* \cup \Omega^{2,n}_*, Z_n, \Omega^1_*)$ are f-closed. Now assume that $|\, u_{\overline{\eta}}(\Omega^1_*) \,| \geq f$. On the one hand, if $|\, u_{\overline{\eta}}(\Omega^1_*) \setminus u_{\overline{\eta}}(Y''_*) \,| > 1$, then there exist distinct m_1, $m_2 \in u_{\overline{\eta}}(\Omega^1_*) \setminus u_{\overline{\eta}}(Y''_*)$ such that $[\, \overline{\eta} \upharpoonright m_1 \,]_{n_1}$, $[\, \overline{\eta} \upharpoonright m_2 \,]_{n_2} \subseteq Y''_*$ for some n_1, $n_2 < \omega$. Without loss of generality, $\overline{\eta} \upharpoonright \langle m_1, n_1 \rangle$, $\overline{\eta} \upharpoonright \langle m_2, n_2 \rangle \in Y''_* \setminus \Omega^1_* \subseteq \bigcup_{n<\omega} \Omega^{2,n}_*$. Thus, $\overline{\eta} \upharpoonright \langle m_1, n_1 \rangle$, $\overline{\eta} \upharpoonright \langle m_2, n_2 \rangle \in \Omega^{2,n}_*$ for some $n < \omega$. Since $(\overline{\eta} \upharpoonright \langle m_1, n_1 \rangle, \overline{\eta} \upharpoonright \langle m_2, n_2 \rangle) \in \mathrm{Cp}(\Omega^{2,n}_*)$ and $\Omega^{2,n}_* \subseteq \mathrm{PC}(\Omega^{2,n}_*, Z_*) \subseteq \Omega^{2,n+1}_*$, it follows that $[\, \overline{\eta} \,]_{N_{\overline{\eta}}} \subseteq [\, \overline{\eta} \,] \cap Z_* \subseteq \mathrm{PC}(\Omega^{2,n}_*, Z_*) \subseteq \Omega^{2,n+1}_* \subseteq Y''_*$. Hence, $\overline{\eta} \in Z''_{Y''_*}$. On the other hand, if $|\, u_{\overline{\eta}}(\Omega^1_*) \setminus u_{\overline{\eta}}(Y''_*) \,| \leq 1$, then $|\, u_{\overline{\eta}}(Y''_*) \,| \geq f-1$. Therefore (Z''_*, Z'', Y''_*) is $(f-1)$-closed.

If $\overline{\eta} \in Y''$, then $|\, u_{\overline{\eta}}(X''_*)\,| < f$ is not possible since that would imply $|\, u_{\overline{\eta}}(X_*)\,| < f$. By the f-closedness of (Z_*, Z, X_*), we would get $\overline{\eta} \in Z_{X_*} \cap Y'' = Y''_{X_*} = \emptyset$. Hence, $|\, u_{\overline{\eta}}(X''_*)\,| \geq f$ for all $\overline{\eta} \in Y''$; and (Y''_*, Y'', X''_*) is f-closed.

Clearly $Y'' \subseteq Z''_{Y''_*}$. If $\overline{\eta} \in Z''_{Y''_*}$, then in particular $\overline{\eta} \in Z'$ or $\overline{\eta} \in Z_n$ for some $n < \omega$. If $|\, u_{\overline{\eta}}(\Omega^1_*)\,| < f$, then $\overline{\eta} \in Z'_{\Omega^1_*} = (Z_n)_{\Omega^1_*} = Y' \subseteq Y''$ by the f-closedness of the triples $(\Omega^1_* \cup \Omega^2_*, Z', \Omega^1_*)$ and $(\Omega^1_* \cup \Omega^{2,n}_*, Z_n, \Omega^1_*)$ for all $n < \omega$. Otherwise, $|\, u_{\overline{\eta}}(\Omega^1_*)\,| \geq f \geq 2$. Hence, there exist $m_1,\, m_2 \in [\,1, k\,]$ such that for all $n < \omega$, $[\,\overline{\eta} \upharpoonright m_1\,]_n$, $[\,\overline{\eta} \upharpoonright m_2\,]_n \nsubseteq \Omega^1_*$. Then we can find n_1, $n_2 < \omega$ such that $\overline{\eta} \upharpoonright \langle m_1, n_1\rangle$, $\overline{\eta} \upharpoonright \langle m_2, n_2\rangle \in \bigcup_{n\in\omega} \Omega^{2,n}_*$. A similar argument as above shows that $[\,\overline{\eta}\,]_{N_{\overline{\eta}}} \subseteq \Omega^{2,n+1}_*$ for some $n < \omega$, showing that $\overline{\eta} \in Z_{\Omega^{2,n+1}_*}$. Observe that $Z'' \subseteq Z \setminus Z_{X_*}$ by the definition of Z'' and (e). It follows that $\overline{\eta} \in Z_{\Omega^{2,n+1}_*} \setminus Z_{X_*} \subseteq Z_{n+1} \subseteq Y''$.

(3) Let $\overline{\eta} \in Z''_{Y''_*} = Y''$. If $\overline{\eta} \in Y'$, then $[\,b_{\overline{\eta}}\,] \subseteq \Omega^1_* \subseteq Y''_*$ since $\mathfrak{F}_{Z_*Z'}$ is (Z', Ω^1_*)-suitable and the families $\mathfrak{F}_{Z_*Z_n}$ are (Z_n, Ω^1_*)-suitable for all $n < \omega$. Now suppose $\overline{\eta} \in Y'' \setminus Y'$. Notice that $|\, u_{\overline{\eta}}(\Omega^1_*)\,| \geq f$ since $|\, u_{\overline{\eta}}(\Omega^1_*)\,| < f$ would imply that for some $n \in \omega$, $\overline{\eta} \in (Z_n)_{\Omega^1_*} = Y'$ by the f-closedness of the triple $(\Omega^1_* \cup \Omega^{2,n}_*, Z_n, \Omega^1_*)$. As before, $\overline{\eta} \in Z_{\Omega^{2,n+1}_*}$, so $[\,b_{\overline{\eta}}\,] \subseteq \Omega^{2,n+1}_* \subseteq Y''_*$ by the $\mathfrak{F}_{Z_*Z}$-closedness of $\Omega^{2,n+1}_*$. This shows that $\mathfrak{F}_{Z_*Z''}$ is (Z'', Y''_*)-suitable.

Since $Y''_{X''_*} = \emptyset$, $\mathfrak{F}_{Z_*Y''}$ is automatically (Y'', X''_*)-suitable.

(4) This follows from Observation 41(3).

(5) Let $\overline{\eta} \in Z''$ and suppose $|\, u_{\overline{\eta}}(X''_*)\,| < f'$. Then $|\, u_{\overline{\eta}}(X_*)\,| < f'$ and so it follows that $\overline{\eta} \in Z_{X_*} \cap Z'' = Z''_{X_*} = \emptyset$, which is a contradiction. Hence, $|\, u_{\overline{\eta}}(X''_*)\,| \geq f'$ for all $\overline{\eta} \in Z''$ and (Z''_*, Z'', X''_*) is f'-closed. $\square$

3. The Step Lemma

DEFINITION 56. Let $0 \le \ell < m \le k$ and $\overline{\xi} \in \Lambda^{[\ell+1,m]}$. Then

$$\Lambda^{\overline{\xi}} = \Lambda^{[1,\ell]} \wedge \{\overline{\xi}\},$$

$$\Lambda_*^{\overline{\xi}} = \Lambda_*^{[1,\ell]} \wedge \{\overline{\xi}\},$$

$$\Lambda^{\overline{\xi}*} = \Lambda^{[1,\ell]} \wedge \left[\,\overline{\xi}\,\right].$$

Furthermore, if $\eta \in {}^{\omega\uparrow}\lambda_\ell$ and $W \subseteq [\,\eta\,]$, let

$$\Lambda_W^{\overline{\xi}} = \Lambda^{[\,1,\ell-1\,]} \wedge W \wedge \{\,\overline{\xi}\,\},$$

$$\Lambda_\eta^{\overline{\xi}*} = \Lambda^{[\,1,\ell-1\,]} \wedge \{\,\eta\,\} \wedge \left[\,\overline{\xi}\,\right].$$

Notice that $\Lambda_*^{\overline{\xi}} \dot\cup \Lambda^{\overline{\xi}*} = [\Lambda^{\overline{\xi}}]$ and $\Lambda_*^{\overline{\xi}} \cap \Lambda^{\overline{\xi}*} = \emptyset$. If $n < \omega$, we write $\Lambda_{\eta\restriction n}^{\overline{\xi}}$ for $\Lambda_{\{\eta\restriction n\}}^{\overline{\xi}}$.

LEMMA 57. *Let $f \in [\,1,k)$, $\overline{\xi} \in \Lambda^{[f+1,k]}$ and $C_* \subseteq \Lambda_*$ be countable such that $\Lambda(C_*) = \emptyset$ (see Definition 45). Then the triple $(C_* \cup [\,\Lambda^{\overline{\xi}}\,], \Lambda^{\overline{\xi}}, C_* \cup \Lambda^{\overline{\xi}*})$ is f-closed.*

PROOF. It is immediate that $(C_* \cup [\Lambda^{\overline{\xi}}], \Lambda^{\overline{\xi}})$ is Λ-closed (with $N_{\overline{\eta}} = 0$ for all $\overline{\eta} \in \Lambda^{\overline{\xi}}$) and that $[\,1,f\,] \subseteq u_{\overline{\eta}}(C_* \cup \Lambda^{\overline{\xi}*})$ because of the properties of C_* and the fact that $\Lambda_*^{\overline{\xi}} \cap \Lambda^{\overline{\xi}*} = \emptyset$. $\square$

DEFINITION 58. (1) For $\overline{\nu} \in \Lambda \cup \Lambda_*$ define $\operatorname{orco} \overline{\nu} = \bigcup_{m \in [\,1,k\,]} \operatorname{Im} \nu_m$ to be the *ordinal content* of $\overline{\nu}$. Furthermore, if $Y_* \subseteq \Lambda \cup \Lambda_*$, then $\operatorname{orco} Y_* = \bigcup_{\overline{\nu} \in Y_*} \operatorname{orco} \overline{\nu}$.

(2) If S, $T \subseteq \lambda_k$ and $\tau : S \to T$ is a bijection, then τ extends canonically to a bijection $\tau : {}^{\omega\geq}S \to {}^{\omega\geq}T$. If $\overline{\nu} \in \Lambda \cup \Lambda_*$ and $\operatorname{orco}\overline{\nu} \subseteq S$, then we define $\overline{\nu}\tau = (\nu_1\tau, \ldots, \nu_k\tau)$.

(3) Given a finite sequence $\langle \lambda'_1, \ldots, \lambda'_k \rangle$ of nonempty subsets of T, consider

$$\Lambda' = {}^{\omega}\lambda'_1 \times \cdots \times {}^{\omega}\lambda'_k,$$

$$\Lambda'_m = \Lambda'^{[1,m-1]} \wedge {}^{\omega>}\lambda'_m \wedge \Lambda'^{[m+1,k]},$$

$$\Lambda'_* = \bigcup_{m\in[1,k]} \Lambda'_m$$

and

$$B' = \bigoplus_{\overline{\nu}\in\Lambda'_*} Ae_{\overline{\nu}},$$

defined in the same way as Λ, Λ_m, Λ_* and B. For $X_* \subseteq \Lambda_*$ such that $\operatorname{orco} X_* \subseteq S$ and $(\operatorname{Im}\nu_m)\tau \subseteq \lambda'_m$ for all $\nu \in X_*$ and $m \in [1,k]$, the bijection $\tau : S \to T$ extends to an A-module monomorphism $\tau : \overline{B}_{X_*} \to \overline{B}'$ called *shift-isomorphism* (onto its image).

(4) For $X_* \subseteq \Lambda_*$, we say that the bijection $\tau : S \to T$ is X_*-*admissible* if $\operatorname{orco} X_* \subseteq S$ and $X_*\tau \subseteq \Lambda_*$.

(5) If $\tau : S \to T$ is an X_*-admissible bijection, then τ extends canonically to an A-module monomorphism $\tau : \overline{B}_{X_*} \to \overline{B}$ called X_*-*admissible shift-isomorphism* (onto its image).

OBSERVATION 59. It is easy to verify that admissible shift-isomorphisms preserve all the notions we have defined so far, namely f-closedness, suitability, pairwise-closedness, etc.

STEP LEMMA 60. *Let the following be given:*

(1) $f \in [0,k)$, $\overline{\xi} \in \Lambda^{[f+1,k]}$ *and a countable* $C_* \subseteq \Lambda_*$ *such that* $\| C_* \| < 0\xi_k$ *and* $\Lambda(C_*) = \emptyset$.

(2) $(J_*, J, I_*) = (J_*(\overline{\xi}), J(\overline{\xi}), I_*(\overline{\xi})) = (C_* \cup [\Lambda^{\overline{\xi}}], \Lambda^{\overline{\xi}}, C_* \cup \Lambda^{\overline{\xi}*})$.

(3) (V_*, V, U_*) *an* $(f+1)$*-closed triple.*

(4) $\mathfrak{F}_{V_*V}$ *regressive and* (V, U_*)*-suitable.*

(5) $\varphi : B_{I_*} \to G_{V_*VU_*}$ *a homomorphism such that* $z\varphi \notin G_{V_*VU_*}$ *for some* $z \in \overline{B}_{C_*}$.

Then, for all $\overline{\eta} \in J$, *we can choose an element* $b_{\overline{\eta}} \in \overline{B}_{C_*}$ *such that the family* $\mathfrak{F}_{J_*J} = \{ x_{\overline{\eta}} = b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in J \}$ *is regressive and* G_{J_*J} *satisfies the following condition:*

If (Z_*, Z, Y_*) *and* (Y_*, Y, X_*) *are two* $(f+1)$*-closed triples such that* $Z_{Y_*} = Y$, $\mathfrak{F}_{Z_*Z} = \{ y'_{\overline{\eta}} = b'_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Z, b'_{\overline{\eta}} \in \overline{B}_{Z_*} \}$ *is regressive and* (Z, Y_*)*-suitable,* $\mathfrak{F}_{Y_*Y} \subseteq \mathfrak{F}_{Z_*Z}$ *is* (Y, X_*)*-suitable and* τ *is a* V_**-admissible bijection such that* $(V_*, V, U_*)\tau = (Y_*, Y, X_*)$ *and* $\mathfrak{F}_{V_*V}\tau = \mathfrak{F}_{Y_*Y}$, *then* $\varphi\tau : B_{I_*} \to G_{Y_*YX_*}$ *does not extend to a homomorphism from* G_{J_*J} *to* $G_{Z_*ZX_*}$.

PROOF. We proceed by induction on f. If $f = 0$, then $\overline{\xi} \in \Lambda$, $J = \{ \overline{\xi} \}$ and $I_* = J_*$ since $\Lambda^{\overline{\xi}}_* = \emptyset$, namely $(J_*, J, I_*) = (C_* \cup [\overline{\xi}], \{ \overline{\xi} \}, C_* \cup [\overline{\xi}])$. Consequently, we only need to define one element $x_{\overline{\xi}}$ by choosing an element $b_{\overline{\xi}}$. We claim that it suffices to take $b_{\overline{\xi}} = \varepsilon z$ for $\varepsilon \in \{0,1\}$. Suppose that there exist two homomorphisms

$$\varphi^{\varepsilon} : \langle B_{J_*}, \varepsilon z + y_{\overline{\xi}} \rangle_* \to G_{V_*VU_*}$$

extending φ. On the one hand, $(z + y_{\overline{\xi}})\varphi^1 - y_{\overline{\xi}}\varphi^0 \in G_{V_*VU_*}$. On the other hand, by continuity of these homomorphisms, $(z + y_{\overline{\xi}})\varphi^1 - y_{\overline{\xi}}\varphi^0 = z\varphi \notin G_{V_*VU_*}$, which is a contradiction. We conveniently choose $b_{\overline{\xi}}$ to satisfy $x_{\overline{\xi}}\varphi \notin G_{V_*VU_*}$. The construction of G_{J_*J} is finished. Moreover, since $G_{V_*VU_*} = G_{V_*V\setminus V_{U_*}U_*}$, this choice does not depend on $\mathfrak{F}_{V_*V_{U_*}}$.

Assume towards a contradiction that there exist two 1-closed triples (Z_*, Z, Y_*) and (Y_*, Y, X_*) with $Z_{Y_*} = Y$, $\mathfrak{F}_{Z_*Z} = \{ y'_{\overline{\eta}} = b'_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in Z, b'_{\overline{\eta}} \in \overline{B}_{Z_*} \}$ regressive and (Z, Y_*)-suitable, $\mathfrak{F}_{Y_*Y} \subseteq \mathfrak{F}_{Z_*Z}$ (Y, X_*)-suitable and a V_*-admissible bijection τ such that $(V_*, V, U_*)\tau = (Y_*, Y, X_*)$, $\mathfrak{F}_{V_*V}\tau = \mathfrak{F}_{Y_*Y}$ and a homomorphism $\psi : G_{J_*J} \to G_{Z_*ZX_*}$ extending $\varphi\tau$. Consider the unique extensions $\widehat{\varphi} : \widehat{G}_{J_*J} \to \widehat{G}_{V_*VU_*}$ and $\widehat{\tau} : \widehat{G}_{V_*VU_*} \to \widehat{G}_{Y_*YX_*}$. Then $\widehat{\varphi}\widehat{\tau} \restriction G_{J_*J} = \psi$ and $x_{\overline{\xi}}\psi \in G_{Z_*ZX_*} \cap \widehat{G}_{Y_*YX_*}$. On the one hand, $\widehat{G}_{Y_*YX_*}/G_{Y_*YX_*}$ is p-divisible. On the other hand, $G_{Z_*ZX_*}/G_{Y_*YX_*} \cong G_{Z_*ZY_*}$ is $\aleph_1$-free by Theorem 42 and Lemma 43. It follows that $x_{\overline{\xi}}\psi = x_{\overline{\xi}}\widehat{\varphi}\widehat{\tau} \in G_{Y_*YX_*}$ and $x_{\overline{\xi}}\widehat{\varphi} \in G_{Y_*YX_*}\tau^{-1} = G_{V_*VU_*}$ contradicting the choice of $b_{\overline{\xi}}$. Therefore, no such ψ exists.

Now assume $f > 0$ and that the statement is true for $f - 1$. Let

$$\lambda_f^o = \{ \alpha < \lambda_f \mid \mathrm{cf}(\alpha) = \omega \},$$

$$\Gamma_\alpha = \{ \eta \in {}^{\omega\uparrow}\lambda_f \mid \sup \eta = \alpha \}$$

and

$$\Gamma = \dot{\bigcup}_{\alpha \in \lambda_f^o} \Gamma_\alpha.$$

Observe that $\Gamma = {}^{\omega\uparrow}\lambda_f$. We well-order each Γ_α to induce a well-order in

$$\Gamma = \{\, \eta^\alpha \mid \alpha < \lambda_f \,\}$$

by lexicographical extension of these orders. For all $\alpha < \lambda_f$, let $\overline{\xi}^\alpha = \eta^\alpha \wedge \overline{\xi} \in \Lambda^{[\,f,k\,]}$. In this way, $\Lambda^{\overline{\xi}} = \bigcup_{\alpha<\lambda_f} \Lambda^{\overline{\xi}^\alpha}$. For $\eta \in {}^{\omega\uparrow}\lambda_f$ and $n < \omega$, let $G_{\eta\restriction n} = B_{\Lambda^{\overline{\xi}}_{[\,\eta\restriction n\,]}}$; and define $\mathcal{G} = \{\, G_{\eta\restriction n} \mid \eta \in {}^{\omega\uparrow}\lambda_f, n < \omega \,\}$. Notice that $G_{\eta\restriction(n+1)} = G_{\eta\restriction n} \oplus B_{\Lambda^{\overline{\xi}}_{\eta\restriction(n+1)}}$.

Take $H_\varphi = \operatorname{Im}\varphi \subseteq G_{V_* V U_*}$ and then construct an $(f+1)$-closed triple $(V'_*, V', U'_*) = (\Omega_*, V', U_* \cap \Omega_*)$ from H_φ and (V_*, V, U_*) (see Observation 52) as described in the proof of Theorem 51 and such that $|\,V'_*\,|$, $|\,V'\,|$, $|\,U'_*\,| \le \lambda_f$, $H_\varphi \subseteq G_{V'_* V' U'_*} \subseteq G_{V_* V U_*}$, $V'_{U_*} = \emptyset$ and $\mathfrak{F}_{V_* V'}$ is (V', U'_*)-suitable. Fix $\Delta \subseteq \lambda_{f+1} \setminus \operatorname{orco} V'_*$ such that $|\,\Delta\,| = \lambda_f$. Define a sequence $\langle\, \lambda'_1, \lambda'_2, \ldots, \lambda'_k \,\rangle$ of sets of ordinals in the following manner: for all $m \in [\,1,k\,]$, let

$$\lambda'_m = \begin{cases} \lambda_m, & \text{if } m \in [\,1,f\,]\,; \\ \\ \Delta \dot\cup \operatorname{orco} V'_*, & \text{if } m \in [\,f+1,k\,]\,. \end{cases}$$

Analogous to the definition of Λ, Λ_m, Λ_* and B, we define

$$\Lambda' = {}^{\omega}\lambda'_1 \times \cdots \times {}^{\omega}\lambda'_k,$$

$$\Lambda'_m = \Lambda'^{[\,1,m-1\,]} \wedge {}^{\omega>}\lambda'_m \wedge \Lambda'^{[\,m+1,k\,]},$$

$$\Lambda'_* = \bigcup_{m\in[\,1,k\,]} \Lambda'_m,$$

$$B' = \bigoplus_{\overline{\nu}\in\Lambda'_*} Ae_{\overline{\nu}}.$$

Let $\mathcal{H} = \{ H \leq \overline{B'} \mid |H| \leq \lambda_{f-1} \}$ and consider the set Θ of all 6-tuples (G, H, P, Q, R, ψ) where $G \in \mathcal{G}$, $H \in \mathcal{H}$, $P \in [\Lambda'_*]^{\leq \lambda_{f-1}}$, $Q \in [\Lambda']^{\leq \lambda_{f-1}}$, $R \in \left[\overline{B'}\right]^{\leq \lambda_{f-1}}$ and $\psi : G \to H$ is a homomorphism. Recall that $\lambda_0 = |A|$ (as agreed on page 21) and notice that

$$|\Theta| = |\mathcal{G}| \cdot |\mathcal{H}| \cdot |\Lambda'_*|^{\lambda_{f-1}} \cdot |\Lambda'|^{\lambda_{f-1}} \cdot \left|\overline{B'}\right|^{\lambda_{f-1}} \cdot \lambda_{f-1}^{\lambda_{f-1}} = \lambda_f^{\lambda_{f-1}} = \lambda_f.$$

Take $\overline{\lambda} = \langle \lambda_f \rangle$ and $\overline{C} = \langle \Theta \rangle$ and apply the $\overline{\lambda}$-Black Box 15 to obtain a family of traps $\langle g_\eta \mid \eta \in \Gamma \rangle = \langle g_{\eta^\alpha} \mid \alpha < \lambda_f \rangle$ with $g_\eta : [\eta] \to \Theta$. For every $\alpha < \lambda_f$, we will choose $b_{\overline{\eta}}$ for all $\overline{\eta} \in \Lambda^{\overline{\xi}^\alpha}$ by examining the trap $g_{\eta^\alpha} : [\eta^\alpha] \to \Theta$ on its values $(\eta^\alpha \upharpoonright n) g_{\eta^\alpha} = (G_{\alpha n}, H_{\alpha n}, P_{\alpha n}, Q_{\alpha n}, R_{\alpha n}, \psi_{\alpha n})$. We will choose these elements by means of the induction hypothesis only under very specific conditions. Otherwise we will take $b_{\overline{\eta}} = 0$ for all $\overline{\eta} \in \Lambda^{\overline{\xi}^\alpha}$. These conditions are:

If there exist $Y, Z \subseteq \Lambda$, $X_*, Y_*, Z_* \subseteq \Lambda_*$, $\mathfrak{F}_{Z_*Z}$, $\Omega^1_* \subseteq Y_*$, $Y' \subseteq Y$, families of subsets $\{ \Omega^{2,n}_* \subseteq Z_* \mid n < \omega \}$ and $\{ Z_n \subseteq Z \mid n < \omega \}$, τ and σ such that

(1) (Z_*, Z, Y_*) and (Y_*, Y, X_*) are $(f+1)$-closed with $Z_{Y_*} = Y$,

(2) $\mathfrak{F}_{Z_*Z}$ is a regressive (Z, Y_*)-suitable family of branch-like elements and $\mathfrak{F}_{Y_*Y} \subseteq \mathfrak{F}_{Z_*Z}$ is (Y, X_*)-suitable,

(3) Ω^1_* is $\mathfrak{F}_{Y_*Y}$-closed and $\Omega^{2,n}_*$ is $\mathfrak{F}_{Z_*Z}$-closed for all $n < \omega$,

(4) $\mathrm{PC}(\Omega^{2,n}_*, Z_*) \subseteq \Omega^{2,n+1}_*$,

(5) $(\Omega^1_*, Y', X_* \cap \Omega^1_*)$ and $(\Omega^1_* \cup \Omega^{2,n}_*, Z_n, \Omega^1_*)$ are $(f+1)$-closed with $(Z_n)_{\Omega^1_*} = Y'$ for all $n < \omega$,

(6) $\mathfrak{F}_{Z_*Y'}$ is $(Y', X_* \cap \Omega^1_*)$-suitable and $\mathfrak{F}_{Z_*Z_n}$ is (Z_n, Ω^1_*)-suitable for all $n < \omega$,

(7) $G_{\Omega^1_* Y' X_* \cap \Omega^1_*} \subseteq G_{Y_* Y X_*}$ and $G_{\Omega^1_* \cup \Omega^{2,n}_* Z_n X_* \cap \Omega^1_*} \subseteq G_{Z_* Z X_*}$ for all $n < \omega$,

(8) $Y'_{X_*} = (Z_n)_{X_*} = \emptyset$ for all $n < \omega$,

(9) $Y' = Z_{\Omega^1_*} \setminus Z_{X_*}$ and $Z_{\Omega^{2,n}_*} \setminus Z_{X_*} \subseteq Z_n \subseteq Z \setminus Z_{X_*}$ for all $n < \omega$,

(10) τ is a V_*-admissible bijection such that

$$(V_*, V, U_*)\tau = (Y_*, Y, X_*),$$

$$(\Omega_*, V', U_* \cap \Omega_*)\tau = (\Omega^1_*, Y', X_* \cap \Omega^1_*),$$

$\mathfrak{F}_{V_* V}\tau = \mathfrak{F}_{Y_* Y}$ and $\mathfrak{F}_{V_* V'}\tau = \mathfrak{F}_{Y_* Y'}$,

(11) $\sigma : \operatorname{orco}(\Omega^1_* \cup \bigcup_{n<\omega} \Omega^{2,n}_*) \to \Delta \cup \operatorname{orco} V'_*$ is a shift-monomorphism such that $\sigma \restriction \operatorname{orco}(\Omega^1_*) = \tau^{-1} \restriction \operatorname{orco}(\Omega^1_*)$,

(12) $G_{\alpha n} = G_{\eta^\alpha \restriction n}$, $H_{\alpha n} \subseteq G_{\Omega^1_* \cup \Omega^{2,n}_* Z_n X_* \cap \Omega^1_*}\sigma$, $P_{\alpha n} = \Omega^{2,n}_*\sigma$, $Q_{\alpha n} = (Z_n \setminus Y')\sigma$, $R_{\alpha n} = \mathfrak{F}_{Z_* Z_n \setminus Y'}\sigma$ and

(13) the maps $\psi_{\alpha n} : G_{\alpha n} \to H_{\alpha n}$ extend each other.

Suppose that at some stage $\alpha < \lambda_f$, we find such $(f+1)$-closed triples $(Z_{\alpha *}, Z_\alpha, Y_{\alpha *})$ and $(Y_{\alpha *}, Y_\alpha, X_{\alpha *})$, a regressive family $\mathfrak{F}_{Z_{\alpha *} Z_\alpha}$ of branch-like elements, $\Omega^1_{\alpha *} \subseteq Y_{\alpha *}$, $Y'_\alpha \subseteq Y_\alpha$, families of subsets $\{\, \Omega^{2,n}_{\alpha *} \subseteq Z_{\alpha *} \mid n < \omega \,\}$ and $\{\, Z_{\alpha n} \subseteq Z_\alpha \mid n < \omega \,\}$ and bijections τ_α and σ_α satisfying these conditions. Define $G_\alpha = \bigcup_{n<\omega} G_{\alpha n} = B_{\Lambda^{\bar{\xi}}_{[\,\eta^\alpha\,]}}$ and $\psi_\alpha = \bigcup_{n<\omega} \psi_{\alpha n} : G_\alpha \to \sum_{n<\omega} H_{\alpha n}$, so

$$\psi_\alpha \sigma_\alpha^{-1} : G_\alpha \to \sum_{n<\omega} G_{\Omega^1_{\alpha *} \cup \Omega^{2,n}_{\alpha *} Z_{\alpha n} X_{\alpha *} \cap \Omega^1_{\alpha *}}.$$

According to Theorem 55 we define an $(f+1)$-closed triple $(Y''_{\alpha *}, Y''_\alpha, X''_{\alpha *})$ as follows:

$$Y''_{\alpha *} = \Omega^1_{\alpha *} \cup \bigcup_{n<\omega} \Omega^{2,n}_{\alpha *},$$

$$Y''_\alpha = Y'_\alpha \cup \bigcup_{n<\omega} Z_{\alpha n},$$

$$X''_{\alpha *} = X_{\alpha *} \cap \Omega^1_{\alpha *}.$$

Consider the triple $(J_*(\overline{\xi}^{\alpha}), J(\overline{\xi}^{\alpha}), I_*(\overline{\xi}^{\alpha})) = (C_* \cup [\,\Lambda^{\overline{\xi}^{\alpha}}\,], \Lambda^{\overline{\xi}^{\alpha}}, C_* \cup \Lambda^{\overline{\xi}^{\alpha}*})$, and notice that

$$B_{I_*(\overline{\xi}^{\alpha})} = B_{C_* \cup \Lambda^{\overline{\xi}^{\alpha}*}} = B_{C_* \cup \Lambda^{\overline{\xi}*}_{\eta^{\alpha}}} \oplus B_{\Lambda^{\overline{\xi}}_{[\,\eta^{\alpha}\,]}} \subseteq B_{I_*(\overline{\xi})} \oplus B_{\Lambda^{\overline{\xi}}_{[\,\eta^{\alpha}\,]}} = B_{I_*(\overline{\xi})} \oplus G_{\alpha}.$$

Let $\varphi'_{\alpha} = \left(\varphi \restriction B_{C_* \cup \Lambda^{\overline{\xi}*}_{\eta^{\alpha}}}\right) \tau_{\alpha} \oplus \psi_{\alpha}\sigma_{\alpha}^{-1}$. Then by definition,

$$\varphi'_{\alpha} : B_{I_*(\overline{\xi}^{\alpha})} \to G_{\Omega^1_{\alpha*} Y'_{\alpha} X_{\alpha*} \cap \Omega^1_{\alpha*}} + \sum_{n<\omega} G_{\Omega^1_{\alpha*} \cup \Omega^{2,n}_{\alpha*} Z_{\alpha n} X_{\alpha*} \cap \Omega^1_{\alpha*}} = G_{Y''_{\alpha*} Y''_{\alpha} X''_{\alpha*}}.$$

We claim that $G_{Y''_{\alpha*} Y''_{\alpha} X''_{\alpha*}} \cap \widehat{G}_{\Omega^1_{\alpha*} Y'_{\alpha} X_{\alpha*} \cap \Omega^1_{\alpha*}} = G_{\Omega^1_{\alpha*} Y'_{\alpha} X_{\alpha*} \cap \Omega^1_{\alpha*}}$. On the one hand, let $g \in G_{Y''_{\alpha*} Y''_{\alpha} X''_{\alpha*}} \cap \widehat{G}_{\Omega^1_{\alpha*} Y'_{\alpha} X_{\alpha*} \cap \Omega^1_{\alpha*}}$, and choose some preimage $g' \in G_{Y''_{\alpha*} Y''_{\alpha}}$ such that $g' \rho_{Y''_{\alpha*} X''_{\alpha*}} = g$. Then we can write

$$g' = b + \sum_{\overline{\eta} \in Y''_{\alpha}} a_{\overline{\eta}} y'_{\overline{\eta}}$$

for some $b \in B_{Y''_{\alpha*}}$ and $a_{\overline{\eta}} \in A$ for all $\overline{\eta} \in Y''_{\alpha}$, where $a_{\overline{\eta}} \neq 0$ only for finitely many elements $\overline{\eta}$. Also notice that $[\,g'\,] \subseteq \Omega^1_{\alpha*}$ and $Y''_{\alpha} \setminus (Y''_{\alpha})_{X''_{\alpha*}} = Y''_{\alpha}$ since $(Y''_{\alpha})_{X''_{\alpha*}} = \emptyset$ by condition (8). Without loss of generality, we can take

$$g' = b + \sum_{\overline{\eta} \in Y''_{\alpha} \setminus Y'_{\alpha}} a_{\overline{\eta}} y'_{\overline{\eta}}$$

(otherwise just reduce g and g'). Take $\overline{\eta} \in [\,g'\,]_{\Lambda} \subseteq Y''_{\alpha} \setminus Y'_{\alpha}$ of maximal norm. Since $[\,g'\,] \subseteq \Omega^1_{\alpha*}$, it follows by condition (9) that

$$\overline{\eta} \in (Y''_{\alpha})_{\Omega^1_{\alpha*}} \setminus Y'_{\alpha} \subseteq (Z_{\alpha} \setminus (Z_{\alpha})_{X_{\alpha*}})_{\Omega^1_{\alpha*}} \setminus Y'_{\alpha} \subseteq ((Z_{\alpha})_{\Omega^1_{\alpha*}} \setminus (Z_{\alpha})_{X_{\alpha*}}) \setminus Y'_{\alpha} = Y'_{\alpha} \setminus Y'_{\alpha} = \emptyset.$$

Therefore, $g' = b \in G_{\Omega^1_{\alpha*} Y'_{\alpha}}$ and $g = g' \rho_{Y''_{\alpha*} X''_{\alpha*}} \in G_{\Omega^1_{\alpha*} Y'_{\alpha} X_{\alpha*} \cap \Omega^1_{\alpha*}}$ showing

$$G_{Y''_{\alpha*} Y''_{\alpha} X''_{\alpha*}} \cap \widehat{G}_{\Omega^1_{\alpha*} Y'_{\alpha} X_{\alpha*} \cap \Omega^1_{\alpha*}} \subseteq G_{\Omega^1_{\alpha*} Y'_{\alpha} X_{\alpha*} \cap \Omega^1_{\alpha*}}.$$

On the other hand, it is immediate that $G_{\Omega^1_{\alpha*}Y'_\alpha X_{\alpha*}\cap\Omega^1_{\alpha*}} \subseteq G_{Y''_{\alpha*}Y''_\alpha X''_{\alpha*}} \cap \widehat{G}_{\Omega^1_{\alpha*}Y'_\alpha X_{\alpha*}\cap\Omega^1_{\alpha*}}$. This proves our claim. Furthermore, $z\varphi'_\alpha = z\varphi\tau_\alpha \notin G_{Y''_{\alpha*}Y''_\alpha X''_{\alpha*}}$ since $z\varphi'_\alpha \in G_{Y''_{\alpha*}Y''_\alpha X''_{\alpha*}}$ would imply $z\varphi'_\alpha \in G_{\Omega^1_{\alpha*}Y'_\alpha X_{\alpha*}\cap\Omega^1_{\alpha*}}$ by our previous claim. Hence, $z\varphi'_\alpha\tau_\alpha^{-1} = z\varphi \in G_{\Omega_* V' U_*\cap\Omega_*} \subseteq G_{V_* V U_*}$, which is a contradiction. In particular, since $(Y''_{\alpha*}, Y''_\alpha, X''_{\alpha*})$ is f-closed and $\overline{\xi}^\alpha \in \Lambda^{[f,k]}$, we can apply the induction hypothesis to choose $b_{\overline{\eta}}$ for all $\overline{\eta} \in \Lambda^{\overline{\xi}^\alpha}$. This finishes the construction of G_{J_*J}.

Assume towards a contradiction that there exist two $(f+1)$-closed triples (Z_*, Z, Y_*) and (Y_*, Y, X_*) with $Z_{Y_*} = Y$, a regressive (Z, Y_*)-suitable family $\mathfrak{F}_{Z_*Z}$ of branch-like elements such that $\mathfrak{F}_{Y_*Y} \subseteq \mathfrak{F}_{Z_*Z}$ is (Y, X_*)-suitable, a V_*-admissible bijection τ such that $(V_*, V, U_*)\tau = (Y_*, Y, X_*)$, $\mathfrak{F}_{V_*V}\tau = \mathfrak{F}_{Y_*Y}$ and a homomorphism $\psi : G_{J_*J} \to G_{Z_*ZX_*}$ extending $\varphi\tau$.

Take $H = \operatorname{Im}\varphi\tau \subseteq G_{Y_*YX_*}$ and $K = \operatorname{Im}\psi \subseteq G_{Z_*ZX_*}$. Apply Theorem 51 to construct the sets Ω^1_*, Y_1, Ω^2_*, Ω^2, Y' and Z' and the two $(f+1)$-closed triples $(\Omega^1_* \cup \Omega^2_*, Z', \Omega^1_*)$ and $(\Omega^1_*, Y', X_* \cap \Omega^1_*)$ such that $H \subseteq G_{\Omega^1_* Y' X_*\cap\Omega^1_*} \subseteq G_{\Omega^1_*\cup\Omega^2_* Z'\Omega^1_*}$ and $K \subseteq G_{\Omega^1_*\cup\Omega^2_* Z'\Omega^1_*} \subseteq G_{Z_*ZX_*}$. Fix an injection $\sigma : \operatorname{orco}(\Omega^1_* \cup \Omega^2_*) \to \Delta \cup \operatorname{orco} V'_*$ such that $\sigma \restriction \operatorname{orco}(\Omega^1_*) = \tau^{-1} \restriction \operatorname{orco}(\Omega^1_*)$.

For all $\eta \in {}^{\omega\uparrow}\lambda_f$ and $n < \omega$, let $K_{\eta\restriction n} = \operatorname{Im}(\psi \restriction G_{\eta\restriction n})$. Put $\Omega^{\eta\restriction 0}_* = \emptyset$; and assume we have constructed $\Omega^{\eta\restriction n}_*$. Let $\Omega^{\eta\restriction n} = Z_{\Omega^{\eta\restriction n}_*}$ and $Z^{\eta\restriction n} = (Y_1 \cup \Omega^{\eta\restriction n}) \setminus (Y_1 \cup \Omega^{\eta\restriction n})_{X_*}$. For every $\overline{\nu} \in \Lambda^{\overline{\xi}}_{\eta\restriction n}$, choose some $h_{\overline{\nu}} \in G_{Z_*Z}$ such that $h_{\overline{\nu}}\rho_{Z_*X_*} = e_{\overline{\nu}}\psi$. Apply Lemma 50 and Theorem 51 to find an $\mathfrak{F}_{Z_*Z}$-closed $\Omega^{\eta\restriction(n+1)}_*$ such that

$$\mathrm{PC}(\Omega^{\eta\restriction n}_*, Z_*) \cup \bigcup_{\overline{\nu}\in\Lambda^{\overline{\xi}}_{\eta\restriction n}} [\, h_{\overline{\nu}} \,] \subseteq \Omega^{\eta\restriction(n+1)}_*.$$

In this way, we obtain families $\{\, \Omega_*^{\eta\restriction n} \subseteq Z_* \mid n < \omega \,\}$ and $\{\, Z^{\eta\restriction n} \subseteq Z \mid n < \omega \,\}$ for all $\eta \in {}^{\omega\uparrow}\lambda_f$ simultaneously.

Let $\mathfrak{g} : {}^{\omega\uparrow>}\lambda_f \to \Theta$ be given by

$$\nu\mathfrak{g} = (G^\nu, H^\nu, P^\nu, Q^\nu, R^\nu, \psi^\nu) = (G_\nu, K_\nu\sigma, \Omega_*^\nu\sigma, (Z^\nu \setminus Y')\sigma, \mathfrak{F}_{Z_* Z^\nu \setminus Y'}\sigma, (\psi \restriction G_\nu)\sigma).$$

By the $\overline{\lambda}$-Black Box 15 there is some $\eta^\alpha \in \Gamma$ such that $g_{\eta^\alpha} \subseteq \mathfrak{g}$. It follows that

$$\begin{aligned}(\eta^\alpha \restriction n)\mathfrak{g} &= (G_{\alpha n}, H_{\alpha n}, P_{\alpha n}, Q_{\alpha n}, R_{\alpha n}, \psi_{\alpha n}) \\ &= (G_{\eta^\alpha\restriction n}, K_{\eta^\alpha\restriction n}\sigma, \Omega_*^{\eta^\alpha\restriction n}\sigma, (Z^{\eta^\alpha\restriction n} \setminus Y')\sigma, \mathfrak{F}_{Z_* Z^{\eta^\alpha\restriction n}\setminus Y'}\sigma, (\psi \restriction G_{\eta^\alpha\restriction n})\sigma).\end{aligned}$$

Consequently, Y, $Z \subseteq \Lambda$, X_*, Y_*, $Z_* \subseteq \Lambda_*$, $\Omega_*^1 \subseteq Y_*$, $Y' \subseteq Y$, $\mathfrak{F}_{Z_*Z}$, τ, σ and the families $\{\, \Omega_*^{\eta^\alpha\restriction n} \subseteq Z_* \mid n < \omega \,\}$ and $\{\, Z^{\eta^\alpha\restriction n} \subseteq Z \mid n < \omega \,\}$ satisfy the specific conditions (1)-(13) of the construction. Let

$Z_*'' = \Omega_*^1 \cup \Omega_*^2 \cup \bigcup_{n<\omega} \Omega_*^{\eta^\alpha\restriction n}$,

$Y_*'' = \Omega_*^1 \cup \bigcup_{n<\omega} \Omega_*^{\eta^\alpha\restriction n}$,

$X_*'' = X_* \cap \Omega_*^1$,

$Z'' = Z' \cup \bigcup_{n<\omega} Z^{\eta^\alpha\restriction n}$ and

$Y'' = Y' \cup \bigcup_{n<\omega} Z^{\eta^\alpha\restriction n}$.

By Theorem 55, (Z_*'', Z'', Y_*'') is f-closed and (Y_*'', Y'', X_*'') is $(f+1)$-closed. Let $\psi_\alpha = \bigcup_{n<\omega} \psi_{\alpha n}$ and

$$\varphi' = \left(\varphi \restriction B_{C_* \cup \Lambda^{\bar{\xi}_*}_{\eta^\alpha}}\right)\tau \oplus \psi_\alpha\sigma^{-1} \subseteq \varphi\tau \oplus (\psi \restriction G_\alpha) \subseteq \psi.$$

Since in the step α of the construction of G_{J_*J} the conditions (1)-(13) for a serious choice of the $b_{\overline{\eta}}$'s were in fact satisfiable, we used some $(f+1)$-closed triples $(Z_{\alpha *}, Z_\alpha, Y_{\alpha *})$ and

$(Y_{\alpha*}, Y_\alpha, X_{\alpha*})$, a regressive family $\mathfrak{F}_{Z_{\alpha*}Z_\alpha}$ of branch-like elements, $\Omega^1_{\alpha*} \subseteq Y_{\alpha*}$, $Y'_\alpha \subseteq Y_\alpha$, families $\{\, \Omega^{2,n}_{\alpha*} \subseteq Z_{\alpha*} \mid n < \omega \,\}$ and $\{\, Z_{\alpha,n} \subseteq Z_\alpha \mid n < \omega \,\}$, τ_α and σ_α in this stage of the construction. Also from this stage, we get an $(f+1)$-closed triple $(Y''_{\alpha*}, Y''_\alpha, X''_{\alpha*})$ and a homomorphism $\varphi'_\alpha = \left(\varphi \restriction B_{C_* \cup \Lambda^{\bar{\xi}_*}_{\eta^\alpha}}\right) \tau_\alpha \oplus \psi_\alpha \sigma_\alpha^{-1}$. Let $\tau' = \sigma_\alpha \sigma^{-1}$. Notice that

$$\Omega^1_{\alpha*}\tau' = (\Omega^1_{\alpha*}\sigma_\alpha)\sigma^{-1} = (\Omega^1_{\alpha*}\tau_\alpha^{-1})\sigma^{-1} = V'_*\sigma^{-1} = \Omega^1_*,$$

$$\Omega^{2,n}_{\alpha*}\tau' = \Omega^{2,n}_{\alpha*}\sigma_\alpha\sigma^{-1} = P_{\alpha n}\sigma^{-1} = \Omega^{\eta^\alpha \restriction n}_*,$$

so $Y''_{\alpha*}\tau' = Y''_*$ and τ' is $Y''_{\alpha*}$-admissible. Furthermore, because of

$$Y'_\alpha\tau' = Y'_\alpha\sigma_\alpha\sigma^{-1} = V'\sigma^{-1} = Y'$$

and

$$(Z_{\alpha,n} \setminus Y'_\alpha)\tau' = Z^{\eta^\alpha \restriction n} \setminus Y',$$

we obtain $Y''_\alpha\tau' = Y''$. Similarly, $X''_\alpha\tau' = X''_*$, which means that $(Y''_{\alpha*}, Y''_\alpha, X''_{\alpha*})\tau' = (Y''_*, Y'', X''_*)$.

It also follows $\mathfrak{F}_{Y''_{\alpha*}Y''_\alpha}\tau' = \mathfrak{F}_{Y''_*Y''}$ and

$$\begin{aligned}
\varphi'_\alpha\tau' &= \left(\varphi \restriction B_{C_* \cup \Lambda^{\bar{\xi}_*}_{\eta^\alpha}}\right) \tau_\alpha\tau' \oplus \psi_\alpha\sigma_\alpha^{-1}\tau' \\
&= \left(\varphi \restriction B_{C_* \cup \Lambda^{\bar{\xi}_*}_{\eta^\alpha}}\right) \tau \oplus \psi_\alpha\sigma^{-1} \\
&= \left(\varphi \restriction B_{C_* \cup \Lambda^{\bar{\xi}_*}_{\eta^\alpha}}\right) \tau \oplus (\psi \restriction G_\alpha) \\
&= \varphi' \subseteq \psi.
\end{aligned}$$

The existence of the f-closed triples (Z''_*, Z'', Y''_*) and (Y''_*, Y'', X''_*) and the fact that $\psi : G_{J_*J} \to G_{Z''_*Z''X''_*}$ extends the homomorphism $\varphi'_\alpha\tau' = \varphi'$ contradicts the previous choice of the elements $\{\, b_{\overline{\eta}} \mid \overline{\eta} \in \Lambda^{\overline{\xi}^\alpha} \,\}$. $\square$

4. The Final Construction

In this last section, we construct $\aleph_k$-free A-modules G with a prescribed group endomorphism ring $\operatorname{End} G = A \oplus \operatorname{Fin} G$. We not only make use of all our previous work, but we also introduce the Strong Black Box principle and a result involving inessential endomorphisms that is based on ideas from Dugas, Göbel [**5**].

DEFINITION 61. $\varphi \in \operatorname{End} G_{Y_*Y}$ is *inessential* if $\overline{B}_{Y_*}\varphi \subseteq G_{Y_*Y}$. We denote the set of inessential endomorphisms of G_{Y_*Y} by $\operatorname{Ines} G_{Y_*Y}$.

LEMMA 62. *Let $\mathcal{V} \subseteq \Lambda$, $\mathfrak{F}_{\Lambda_*\mathcal{V}}$ be a regressive family of branch-like elements and $G = G_{\Lambda_*\mathcal{V}}$. If $\varphi \in \operatorname{End} G \setminus (A \oplus \operatorname{Ines} G)$, then there exist a countably-generated A-submodule P of B and some $z \in \overline{P}$ such that $z\varphi \notin \langle G, Az\rangle_*$.*

PROOF. Let us begin with the following observation: if $g \in G$ has infinite support and if $\overline{\eta} \in [\,g\,]_\Lambda$ is of maximal norm, then $\overline{\eta} \in \Lambda([\,g\,])$. In particular, $\Lambda([\,g\,]) \neq \emptyset$.

Let $\mathcal{D} = \{\,(n,a) \mid n = 0 \text{ or } a \notin p^n A\,\} \subseteq \omega \times A$. Our first goal is to find a countably-generated A-submodule P of B such that $\overline{P}(p^n\varphi - a) \nsubseteq G$ for all $(n,a) \in \mathcal{D}$. Let us start by finding a countably-generated A-submodule P' of B such that $\overline{P'}(\varphi - a) \nsubseteq G$ for all $a \in A$, i.e. for all $(0,a) \in \mathcal{D}$. Choose some $w \in \overline{B} \setminus B$ with $\Lambda([\,w\,]) = \emptyset$; and let $P_0 = B_{[\,w\,]}$. Suppose that $P' = P_0$ fails to have the desired property. Then, $\overline{P_0}(\varphi - a) \subseteq G$ for some $a \in A$. Since $\varphi - a \notin \operatorname{Ines} G$, there exists some $w' \in \overline{B}$ such that $w'(\varphi - a) \notin G$. Let $P_1 = B_{[\,w\,]\cup[\,w'\,]}$. Towards a contradiction, suppose again that $\overline{P_1}(\varphi - a') \subseteq G$ for some $a' \in A$. Since $w \in \overline{P_0} \subseteq \overline{P_1}$, we have $w(\varphi - a) - w(\varphi - a') =$

$w(a'-a) \in G$ which is impossible for $a \neq a'$ because $\Lambda([\,w\,]) = \emptyset$, so $w(a'-a)$ cannot be an element of G. This forces $a = a'$, from which we get the desired contradiction

$$G \ni w'(\varphi - a') = w'(\varphi - a) \notin G.$$

Hence, it suffices to take $P' = P_0$ or $P' = P_1$.

Our next task is to find a countably-generated submodule P of B containing P' with the required property. Let

$$\mathcal{N} = \{\, n < \omega \mid \exists a_n \in A\,(\,(n, a_n) \in \mathcal{D} \text{ and } \overline{P'}(p^n\varphi - a_n) \subseteq G\,)\,\}.$$

Observe that a_n is unique for $n \in \mathcal{N}$: if (n, a_n), $(n, a'_n) \in \mathcal{D}$, then

$$w(p^n\varphi - a'_n) - w(p^n\varphi - a_n) = w(a_n - a'_n) \in G,$$

and our previous argument implies that $a_n = a'_n$.

Consider $n \in \mathcal{N}$ and put $\psi = p^n\varphi - a_n$. We recursively construct a family $C = \{\, \overline{\nu}_s \mid s < \omega \,\} \subseteq \Lambda_*$ such that $\Lambda(C) = \emptyset$, $\max_{r<s} \|e_{\overline{\nu}_r}\psi\| < \|e_{\overline{\nu}_s}\|$ and $C \cap [\,\overline{\eta}\,]$ is finite for all $\overline{\eta} \in \Lambda$. Let

$$x_n = \sum_{r=0}^{\infty} p^{(n+1)r} e_{\overline{\nu}_r}.$$

Notice that

$$\sum_{r=s+1}^{\infty} p^{(n+1)r} e_{\overline{\nu}_r} = p^{(n+1)s+n} \sum_{r=1}^{\infty} p^{(n+1)r-n} e_{\overline{\nu}_{s+r}},$$

for every $s < \omega$; and

$$\begin{aligned} x_n\psi &= \left(\sum_{r=0}^{s-1} p^{(n+1)r} e_{\overline{\nu}_r}\psi\right) + p^{(n+1)s} e_{\overline{\nu}_s}(p^n\varphi - a_n) + \left(\sum_{r=s+1}^{\infty} p^{(n+1)r} e_{\overline{\nu}_r}\psi\right) \\ &= \left(\sum_{r=0}^{s-1} p^{(n+1)r} e_{\overline{\nu}_r}\psi\right) - p^{(n+1)s} a_n e_{\overline{\nu}_s} + p^{(n+1)s+n}\left(e_{\overline{\nu}_s}\varphi + \sum_{r=1}^{\infty} p^{(n+1)r-n} e_{\overline{\nu}_{s+r}}\psi\right). \end{aligned}$$

It follows that $C \subseteq [x_n\psi]$ since $\max_{r<s} \| e_{\overline{\nu}_r}\psi \| < \| e_{\overline{\nu}_s} \|$ and $a_n \notin p^n A$. Therefore, $x_n\psi \notin G$. We do this construction for every $n \in \mathcal{N}$. Let $P = B_{S\cup[\,P'\,]}$ where $S = \bigcup_{n\in\mathcal{N}}[x_n]$. To verify that P has the required property, suppose $\overline{P}(p^n\varphi - a) \subseteq G$ for some $(n,a) \in \mathcal{D}$. This means that $n \in \mathcal{N}$ and $a = a_n$, but then again

$$G \ni x_n(p^n\varphi - a) = x_n(p^n\varphi - a_n) \notin G$$

yields a contradiction. Hence P is as required.

Now take some $x \in \overline{B} \setminus B$ such that $[\,x\,] \cap ([\,P\,] \cup [\,P\varphi\,]) = \emptyset$ and $[\,x\,] \cap [\,\overline{\eta}\,]$ is finite for all $\overline{\eta} \in \Lambda$. If $x\varphi \in \langle\, G, Ax \,\rangle_*$, then

$$x(p^n\varphi - a) \in G \tag{1}$$

for some $n < \omega$ and $a \in A$. Since G is pure, we can assume $(n,a) \in \mathcal{D}$. Take $x' \in \overline{P}$ such that $x'(p^n\varphi - a) \notin G$. If we assume $(x + x')\varphi \in \langle\, G, A(x + x')\,\rangle_*$, then

$$(x + x')(p^m\varphi - a') \in G \tag{2}$$

for some $m \in [\,n,\omega)$ and $a' \in A$. Substracting (1) multiplied by p^{m-n} from (2), we obtain

$$(x + x')(p^m\varphi - a') - p^{m-n}x(p^n\varphi - a) = x'(p^m\varphi - a') + x(p^{m-n}a - a') \in G.$$

This yields $p^{m-n}a = a'$ since $[\,x\,] \cap ([\,P\,] \cup [\,P\varphi\,]) = \emptyset$ and $[\,x\,] \cap [\,\overline{\eta}\,]$ is finite for all $\overline{\eta} \in \Lambda$. But then, $x'(p^m\varphi - a') \in G$ which is a contradiction. Hence we can take $z = x$ or $z = x + x'$. $\square$

For a group G, define $\operatorname{Fin} G$ to the set of endomorphisms of G with finite rank. Because A is slender, $\operatorname{Ines} G = \operatorname{Fin} G$ for the kind of groups we have constructed. We refer the reader to Göbel, Trlifaj [**14**], Theorem 12.3.37 for a detailed proof.

We now present the Strong Black Box, which is the last prediction principle that will be used in the final construction. See Göbel, Trlifaj [**14**] for a proof.

Definition 63. A *trap* for the Strong Black Box is a quintuple $p = (\eta, V_*, V, \mathfrak{F}, \varphi)$ such that

(1) $\eta \in {}^{\omega\uparrow}\lambda_k$,

(2) $V \in [\,\Lambda\,]^{\leq \lambda_{k-1}}$ and $V_* \in [\,\Lambda_*\,]^{\leq \lambda_{k-1}}$,

(3) (V_*, V) is Λ-closed,

(4) $\Lambda^{\eta *} \subseteq V_*$,

(5) $\|\,\overline{\xi}\,\| < \|\,\eta\,\|$ for all $\overline{\xi} \in V \cup V_*$,

(6) For $\overline{\eta} \in \Lambda$, if $\|\,\overline{\eta}\,\| < \|\,\eta\,\|$ and $k \notin u_{\overline{\eta}}(V_*)$, then $\overline{\eta} \in V$.

(7) For $\overline{\eta} \in \Lambda$, if $([\,\overline{\eta}\,] \setminus [\,\overline{\eta} \upharpoonright k\,]) \cap V_* \neq \emptyset$, then $[\,\overline{\eta}\,] \subseteq V_*$.

(8) $\mathfrak{F} = \mathfrak{F}_{V_*V} = \{\, y'_{\overline{\eta}} = b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in V, b_{\overline{\eta}} \in \overline{B}_{V_*} \,\}$ is regressive,

(9) $\varphi : G_{V_*V} \to G_{V_*V}$ is a homomorphism.

The Strong Black Box 64. *Let μ be an infinite cardinal, $\lambda = \mu^+$, $\theta \leq \lambda$ such that $\mu^\theta = \mu$ and $k > 1$. If $E \subseteq \lambda^o$ is stationary, then there is a family*

$$\{\, p_\alpha = (\eta^\alpha, V_{\alpha *}, V_\alpha, \mathfrak{F}_\alpha, \varphi_\alpha) \mid \alpha < \lambda \,\}$$

of traps such that

(1) $\|\,\eta^\alpha\,\| \in E$ *for all* $\alpha < \lambda$,

(2) $\| \eta^\alpha \| \leq \left\| \eta^\beta \right\|$ *for all* $\alpha < \beta < \lambda$,

(3) *If* $\| \eta^\alpha \| = \left\| \eta^\beta \right\|$ *for* $\alpha \neq \beta$, *then* $\| V_{\alpha *} \cap V_{\beta *} \| < \| \eta^\alpha \|$,

(4) *For any* $\mathcal{V} \subseteq \Lambda$, *any regressive family* $\mathfrak{F}_{\Lambda_* \mathcal{V}} = \{ y'_{\overline{\eta}} = b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in \mathcal{V}, b_{\overline{\eta}} \in \overline{B} \}$, *any* $\varphi \in \operatorname{End} G_{\Lambda_* \mathcal{V}}$, $U \in [\Lambda_*]^{\leq \theta}$ *and* $\delta < \lambda$, *the set*

$$\{ \gamma \in E \mid \exists \alpha < \lambda \; (\| \eta^\alpha \| = \gamma, \, \delta < 0\eta^\alpha, \, V_\alpha = \mathcal{V}_{V_{\alpha *}}, \, \mathfrak{F}_\alpha = \mathfrak{F}_{\Lambda_* V_\alpha}, \, \varphi_\alpha \subseteq \varphi, \, U \subseteq V_{\alpha *}) \}$$

is unbounded, where $V_\alpha = \mathcal{V}_{V_{\alpha *}}$ *is only possible for* $k > 1$.

Main Theorem 65. *If A is a ring with free additive group $A^+ = \bigoplus_{\alpha < \kappa} \mathbb{Z} e_\alpha$ such that $\overline{A} = \widehat{A} \cap \prod_{\alpha < \kappa} \mathbb{Z} e_\alpha$ is an A-module, $|A| < \mu$, $k \in (0, \omega)$ and $\lambda = \beth_k^+(\mu)$, then it is possible to construct an $\aleph_k$-free A-module G of cardinality λ such that G is separable as an abelian group and* $\operatorname{End} G = A \oplus \operatorname{Ines} G = A \oplus \operatorname{Fin} G$.

Proof. Since the case $k = 1$ is a classical result due to M. Dugas and R. Göbel (see [**5**]), we assume $k > 1$. Consider the stationary subset $\lambda_k^o = \{ \alpha < \lambda_k \mid \operatorname{cf}(\alpha) = \omega \}$ of $\lambda_k = \mu_k^+$. By Solovay's Theorem we can decompose λ_k^o into λ_k disjoint stationary subsets, say $\lambda_k^o = \dot{\bigcup}_{\alpha < \lambda_k} E_\alpha$. Now let

$$\mathcal{I} = \{ (P, z) \mid P \leq B \text{ is countably-generated}, z \in \overline{P} \setminus P \},$$

and notice that $|\mathcal{I}| = \lambda_k$. Therefore, we are allowed to write

$$\lambda_k^o = \dot{\bigcup_{(P,z) \in \mathcal{I}}} E_{(P,z)}.$$

For each $E_{(P,z)}$, the Strong Black Box provides us with a family of traps

$$p_\alpha^{(P,z)} = (\eta^{\alpha\, (P,z)}, V_{\alpha *}^{(P,z)}, V_\alpha^{(P,z)}, \mathfrak{F}_\alpha^{(P,z)}, \varphi_\alpha^{(P,z)})$$

for $\alpha < \lambda_k$. We gather all these traps and order them according to the norm of their first component, namely $\| \eta^\alpha \| \leq \left\| \eta^\beta \right\|$ for all $\alpha < \beta < \lambda_k$.

Let $\mathcal{V} = \bigcup_{\alpha<\lambda_k} \Lambda^{\eta^\alpha}$ (see Definition 56). We will construct a regressive family $\mathfrak{F}_{\Lambda_*\mathcal{V}}$ of branch-like elements by choosing for all $\alpha < \lambda_k$ and all $\overline{\eta} \in \Lambda^{\eta^\alpha}$ an element $b_{\overline{\eta}} \in \overline{B}$, and define $G = G_{\Lambda_*\mathcal{V}}$. If we get, when considering the trap $p_\alpha = (\eta^\alpha, V_{\alpha*}, V_\alpha, \mathfrak{F}_\alpha, \varphi_\alpha)$ and the unique $(P, z) \in \mathcal{I}$ such that $\| \eta^\alpha \| \in E_{(P,z)}$, that $[\, P\,] \subseteq V_{\alpha*}$, $\| z \| < 0\eta^\alpha$ and $z\varphi_\alpha \notin G_{V_{\alpha*}V_\alpha}$, we will obtain the $b_{\overline{\eta}}$'s for $\overline{\eta} \in \Lambda^{\eta^\alpha}$ by means of the Step Lemma 60. Otherwise we will set $b_{\overline{\eta}} = 0$ for all $\overline{\eta} \in \Lambda^{\eta^\alpha}$.

Let $\alpha < \lambda_k$ and (P, z) be the unique element of $\mathcal{I}$ such that $\| \eta^\alpha \| \in E_{(P,z)}$. Assume $[\, P\,] \subseteq V_{\alpha*}$, $\| z \| < 0\eta^\alpha$ and $z\varphi_\alpha \notin G_{V_{\alpha*}V_\alpha}$. Choose two elements z_1, $z_2 \in P$ with infinite support such that $z_1 + z_2 = z$, $[\, z_1\,] \dot\cup [\, z_2\,] = [\, z\,]$ and $u_{\overline{\eta}}([\, z_1\,]) = u_{\overline{\eta}}([\, z_2\,]) = [\, 1, k\,]$ for all $\overline{\eta} \in \Lambda$. Without loss of generality, $z_1\varphi_\alpha \notin G_{V_{\alpha*}V_\alpha}$. Let $V_* = V_{\alpha*} \dot\cup \Lambda_*^{\eta^\alpha}$, $V = V_\alpha \dot\cup \Lambda^{\eta^\alpha}$ and $U_* = \left[\, \Lambda^{\eta^\alpha} \,\right] \cup [\, z_1\,]$. These unions are indeed disjoint because $V_{\alpha*} \cap \Lambda_*^{\eta^\alpha} = V_\alpha \cap \Lambda^{\eta^\alpha} = \emptyset$ by Definition 63(5).

We claim that (V_*, V, U_*) is k-closed: Since $(V_{\alpha*}, V_\alpha)$ is Λ-closed and $\Lambda^{\eta^\alpha *} \subseteq V_{\alpha*}$, it follows that (V_*, V) is Λ-closed. If $\overline{\eta} \in V$, then either $\overline{\eta} \in V_\alpha$ or $\overline{\eta} \in \Lambda^{\eta^\alpha}$. If $\overline{\eta} \in V_\alpha$, then in particular $\eta_k \neq \eta^\alpha$ since $\| \overline{\eta} \| < \| \eta^\alpha \|$, so $[\, \overline{\eta}\,]_n \cap \left[\, \Lambda^{\eta^\alpha} \,\right] = \emptyset$ for all large enough $n < \omega$. Moreover, for all $m \in [\, 1, k\,]$, $[\, \overline{\eta} \restriction m\,]_n \nsubseteq [\, z_1\,]$ since $u_{\overline{\eta}}([\, z_1\,]) = [\, 1, k\,]$. Hence, $u_{\overline{\eta}}(U_*) = [\, 1, k\,]$ for all $\overline{\eta} \in V_\alpha$. If $\overline{\eta} \in \Lambda^{\eta^\alpha}$, then $[\overline{\eta}] \subseteq \left[\, \Lambda^{\eta^\alpha} \,\right]$ and $\overline{\eta} \in V_{U_*}$. This proves our claim. Moreover, $\Lambda^{\eta^\alpha} = V_{U_*}$.

Let $\mathfrak{F}_{V_*V} = \mathfrak{F}_\alpha \cup \mathfrak{F}_{V_*V_{U_*}}$ where $\mathfrak{F}_{V_*V_{U_*}} = \{\, \varepsilon_{\overline{\eta}} z_1 + y_{\overline{\eta}} \mid \overline{\eta} \in \Lambda^{\eta^\alpha} \,\}$ for some $\varepsilon_{\overline{\eta}} \in \{\, 0, 1\,\}$. Observe that if $\overline{\eta} \in V_{U_*}$, then either $b_{\overline{\eta}} = 0$ or $[\, b_{\overline{\eta}}\,] = [\, z_1\,] \subseteq U_*$. Hence, $\mathfrak{F}_{V_*V}$ is

(V, U_*)-suitable. Put $f = k-1$, $\overline{\xi} = \eta^\alpha$, $C_* = [\, z_1 \,]$ and

$$\psi_\alpha = (\varphi_\alpha \restriction B_{I_*(\overline{\xi})})\rho_{V_* U_*} : B_{I_*(\overline{\xi})} \to G_{V_* V U_*}$$

(recall that $(J_*(\overline{\xi}), J(\overline{\xi}), I_*(\overline{\xi})) = (C_* \cup [\, \Lambda^{\overline{\xi}} \,], \Lambda^{\overline{\xi}}, C_* \cup \Lambda^{\overline{\xi}*})$ as in the Step Lemma 60), and notice $z_1\psi_\alpha \notin G_{V_* V U_*}$ since $z_1\varphi_\alpha \notin G_{V_{\alpha*} V_\alpha}$. We now choose the elements $b_{\overline{\eta}} \in \overline{B}_{J_*}$ for all $\overline{\eta} \in J = \Lambda^{\eta^\alpha}$ by applying the Step Lemma 60 (remember that the choice of these elements is independent from $\mathfrak{F}_{V_* V_{U_*}}$).

To obtain a contradiction, suppose that there is some $\varphi \in \operatorname{End} G \setminus (A \oplus \operatorname{Ines} G)$. We apply Lemma 62 to G to obtain a countably-generated submodule P of B and some $z \in \overline{P}$ such that $z\varphi \notin \langle G, Az \rangle_*$. Then, $z \in \overline{P} \setminus P$; and (P, z) is an element of the index set $\mathcal{I}$. Since the set

$$\{\, \gamma \in E_{(P,z)} \mid \exists \alpha < \lambda_k \,(\, \left\| \eta^{\alpha\,(P,z)} \right\| = \gamma,\ \|\, z \,\| < 0\eta^{\alpha\,(P,z)},\ V_\alpha^{(P,z)} = \mathcal{V}_{V_{\alpha*}^{(P,z)}},$$

$$\mathfrak{F}_\alpha^{(P,z)} = \mathfrak{F}_{\Lambda_* V_\alpha^{(P,z)}},\ \varphi_\alpha^{(P,z)} \subseteq \varphi,\ [P] \subseteq V_{\alpha*}^{(P,z)}) \,\}$$

is unbounded by the Strong Black Box, we can choose an ordinal γ from this set with the corresponding trap $p_{\alpha'}^{(P,z)}$. Pick $\alpha < \lambda_k$ such that $p_{\alpha'}^{(P,z)} = p_\alpha$. Because of $[\, P \,] \subseteq V_{\alpha*}$, $\|\, z \,\| < 0\eta^\alpha$ and $z\varphi_\alpha \notin G_{V_{\alpha*} V_\alpha} \subseteq G$, the conditions for applying the Step Lemma 60 are satisfied. Take z_1 and z_2 as before. Let $(Z_*, Z) = (\Lambda_*, \mathcal{V})$, which is clearly Λ-closed, and $(Y_*, Y, X_*) = (V_*, V, U_*)$. We have already seen that (Y_*, Y, X_*) is k-closed. To prove that (Z_*, Z, Y_*) is also k-closed, let $\overline{\eta} \in Z = \bigcup_{\alpha < \lambda_k} \Lambda^{\eta^\alpha}$. Thus, $\eta_k = \eta^\beta$ for some $\beta < \lambda_k$. If $\left\| \eta^\beta \right\| > \|\, \eta^\alpha \,\|$, then $u_{\overline{\eta}}(Y_*) = [\, 1, k \,]$ since $\|\, Y_* \,\| = \|\, \eta^\alpha \,\|$. Now assume $\left\| \eta^\beta \right\| < \|\, \eta^\alpha \,\|$ and $|\, u_{\overline{\eta}}(Y_*) \,| < k$. Notice that, in this case, $u_{\overline{\eta}}(Y_*) = u_{\overline{\eta}}(V_{\alpha*})$. If

$k \notin u_{\overline{\eta}}(Y_*)$, then $\overline{\eta} \in V_\alpha = (\mathcal{V})_{V_{\alpha*}} \subseteq (\mathcal{V})_{Y_*}$ by Definition 63(6) and the Λ-closedness of $(V_{\alpha*}, V_\alpha)$. Hence, $\overline{\eta} \in Z_{Y_*}$. If $m \notin u_{\overline{\eta}}(Y_*)$ for some $m \neq k$, then $\overline{\eta} \upharpoonleft \langle m, n\rangle \in V_{\alpha*}$ for some $n < \omega$, which implies that $[\,\overline{\eta}\,] \subseteq V_{\alpha*}$ by Definition 63(7). Consequently, $\overline{\eta} \in Z_{Y_*}$ as well. Finally, assume $\|\,\eta^\beta\,\| = \|\,\eta^\alpha\,\|$. If $\beta = \alpha$, then $[\,\overline{\eta}\,] \subseteq Y_*$, so $\overline{\eta} \in Z_{Y_*}$. If $\beta \neq \alpha$, then $\overline{\eta} \upharpoonleft \langle m, n\rangle \notin Y_*$ for all $m \neq k$ and $n < \omega$ since $\|\,\overline{\eta} \upharpoonleft \langle m, n\rangle\,\| = \|\,\eta^\alpha\,\|$. This implies $[\,1, k-1\,] \subseteq u_{\overline{\eta}}(Y_*)$. If $k \notin u_{\overline{\eta}}(Y_*)$, then $[\,\overline{\eta} \upharpoonleft k\,]_n \subseteq V_{\alpha*}$ for some $n < \omega$. By Definition 63(4), we also have $[\,\overline{\eta} \upharpoonleft k\,]_n \subseteq V_{\beta*}$. This yields $[\,\overline{\eta} \upharpoonleft k\,]_n \subseteq V_{\alpha*} \cap V_{\beta*}$ and $\|\,V_{\alpha*} \cap V_{\beta*}\,\| = \|\,\eta^\alpha\,\|$ contradicting the Strong Black Box 64. Therefore, $u_{\overline{\eta}}(Y_*) = [\,1, k\,]$. This proves that (Z_*, Z, Y_*) is k-closed.

Now we verify $Z_{Y_*} = Y$. It is immediate that $Y \subseteq Z_{Y_*}$. Let $\overline{\eta} \in Z_{Y_*}$ with $\eta_k = \eta^\beta$ for some $\beta < \lambda_k$. If $\beta = \alpha$, then $\overline{\eta} \in \Lambda^{\eta^\alpha} \subseteq Y$. If $\beta \neq \alpha$, then $[\,\overline{\eta}\,]_n \subseteq V_{\alpha*}$ for some $n < \omega$. In particular $\|\,\overline{\eta}\,\| < \|\,\eta^\alpha\,\|$ and $u_{\overline{\eta}}(V_{\alpha*}) = \emptyset$. By Definition 63(6), $\overline{\eta} \in V_\alpha$, so $\overline{\eta} \in Y$ and $Z_{Y_*} \subseteq Y$.

Finally, we check that $\mathfrak{F}_{\Lambda_*\mathcal{V}}$ is (Z, Y_*)-suitable and $\mathfrak{F}_{\Lambda_* Y}$ is (Y, X_*)-suitable. If $\overline{\eta} \in Z_{Y_*} = Y$, then $\overline{\eta} \in V_\alpha$ or $\overline{\eta} \in \Lambda^{\eta^\alpha}$. In the first case, $b_{\overline{\eta}} \in \overline{B}_{V_{\alpha*}}$ by the definition of $\mathfrak{F}_\alpha$. Thus, $[\,b_{\overline{\eta}}\,] \subseteq V_{\alpha*}$. In the second case, $b_{\overline{\eta}} = z_1$; and $[\,b_{\overline{\eta}}\,] \subseteq X_* \subseteq Y_*$ follows. This proves that $\mathfrak{F}_{\Lambda_*\mathcal{V}}$ is (Z, Y_*)-suitable. Moreover, the second case also yields that $\mathfrak{F}_{\Lambda_* Y}$ is (Y, X_*)-suitable since $Y_{X_*} = V_{U_*} = \Lambda^{\eta^\alpha}$. Hence, the homomorphism $\varphi\rho_{Z_*X_*} = \varphi\rho_{\Lambda_*U_*}$ extends the homomorphism

$$\psi_\alpha = (\varphi_\alpha \restriction B_{I_*(\overline{\xi})})\rho_{Y_*X_*} = (\varphi_\alpha \restriction B_{I_*(\eta^\alpha)})\rho_{V_*U_*}.$$

The existence of (Z_*, Z, Y_*), (Y_*, Y, X_*), $\mathfrak{F}_{\Lambda_*\mathcal{V}}$, $\tau = \mathrm{Id}$ and $\psi_\alpha \subseteq \varphi\rho_{Z_*X_*}$ contradicts the choice of the $b_{\overline{\eta}}$'s for $f = k-1$, $\overline{\xi} = \eta^\alpha$, $C_* = [\,z_1\,]$ and z_1. $\square$

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